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Identities for the fundamental solution of thin plate bending problems and the nonuniqueness of the hypersingular BIE solution for multi-connected domains

Y.J. Liu *, S. Huang

Mechanical Engineering, University of Cincinnati, P.O. Box 210072, Cincinnati, OH 45221-0072, USA

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A B S T R A C T

Four integral identities for the fundamental solution of thin plate bending problems are presented in this paper. These identities can be derived by imposing rigid-body translation and rotation solutions to the two direct boundary integral equations (BIEs) for plate bending problems, or by integrating directly the governing equation for the fundamental solution. These integral identities can be used to develop weakly-singular and nonsingular forms of the BIEs for plate bending problems. They can also be employed to show the nonuniqueness of the solution of the hypersingular BIE for plates on multi-connected (or multiply-connected) domains. This nonuniqueness is shown for the first time in this paper. It is shown that the solution of the singular (deflection) BIE is unique, while the hypersingular (rotation) BIE can admit an arbitrary rigid-body translation term in the deflection solution, on the edge of a hole. However, since both the singular and hypersingular BIEs are required in solving a plate bending problem using the boundary element method (BEM), the BEM solution is always unique on edges of holes in plates on multi-connected domains. Numerical examples of plates with holes are presented to show the correctness and effectiveness of the BEM for multi-connected domain problems.

1. Introduction

The boundary element method (BEM) has been applied successfully to solve the thin plate bending problem since the late 1970s and early 1980s based on the direct boundary integral equation (BIE) formulations for both linear and nonlinear responses [1–11]. The governing equation for the small deflection of a thin elastic plate is the biharmonic equation in terms of the deflection of the plate under a lateral load. Using the Rayleigh-Green identity and the fundamental solution, the biharmonic governing equation can be transformed into a direct BIE formulation. In this direct BIE formulation, there are four boundary variables, that is, the deflection, rotation angle, bending moment, and the Kirchhoff equivalent shear force. Two of these boundary variables are given from the boundary conditions (BCs) and the other two need to be determined. Therefore, two BIEs are required in the BIE formulation for the thin plate bending problem: the displacement (deflection) BIE and the rotation (normal derivative) BIE. The first BIE is strongly singular, whereas the second is hypersingular. All these characteristics resemble those of the BIE formulations for potential, elasticity, Stokes flow, acoustic and elastodynamic problems, except for the fact that the use of the hypersingular BIE together with the singular BIE is a must for solving the plate bending problem with the BEM.

For multi-connected (or multiply-connected) domain elasticity and Stokes flow problems, it has been shown that the hypersingular or traction BIEs have nonunique solutions on the boundary of a hole or void where no constraint is applied. That is, a constant translation term can be added to the solution of the displacement on the boundary of the hole or void and the hypersingular BIE still holds. This is because of the properties of the hypersingular kernels, or the identities satisfied by such kernels [12–14]. This defect in the hypersingular BIEs for multi-connected domains was reported in Refs. [15,16] for elasticity problems and in Refs. [17,18] for Stokes flow problems. Special care or techniques were also proposed in Refs. [15,16] to remedy this situation with the traction BIE for multi-connected domain elasticity problems. For Stokes flow problems, a dual BIE formulation using a linear combination of the singular and hypersingular BIEs is suggested in Refs. [17,18].

However, the above mentioned nonuniqueness solution problem has not been reported in the literature for the thin plate bending BIE formulation for multi-connected domains. To apply the BEM to solve more practical and large-scale plate bending problems, such as the study of the bending behaviors of perforated plates (plates with numerous holes), the possible nonuniqueness problem of the direct BIE formulation for thin plate bending problems must be addressed.

It should be pointed out that this nonuniqueness problem with the hypersingular BIE formulations for either elasticity or plate

* Corresponding author.
E-mail address: Yijun.Liu@uc.edu (Y.J. Liu).

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bending problems on multi-connected domains are general, in the sense that it exists regardless of the sizes and shapes of the holes or voids, or in 2-D or 3-D domains. For the specific degenerate scale problems of the BIEs for multi-connected domains, Refs. [19–23] can be consulted, which address the nonuniqueness of the BIE solutions for problems in a domain with a specific dimension, size or shape.

In this paper, the issue of the nonuniqueness solution with the direct BIE formulation for thin plate bending problems on multi-connected domains is studied. To do this, four integral identities for the fundamental solution of plate bending problems are derived first. These identities are derived by imposing the rigid-body translation and rotation solutions to the direct BIEs and by directly integrating the governing equation of the fundamental solution. Then, it is shown that the singular BIE deflection solution is unique while the hypersingular BIE deflection solution is not unique on the edge of a hole. That is, the solution of the hypersingular BIE can have an arbitrary rigid-body translation term. These results are consistent with those for the BIEs for multi-connected domain elasticity problems. However, these observations have not been reported in the literature, to the best knowledge of the authors.

The remaining of this paper is organized as following: in Section 2, the direct BIE formulations for thin plate bending problems are reviewed. Key results scattered in the literature are summarized. In Section 3, four integral identities are derived using the two methods mentioned in the above. In Section 4, the nonuniqueness issue of the deflection solutions of the thin plate bending BIEs for multi-connected domain problems is investigated using the derived integral identities. In Section 5, a few numerical examples of plates with holes and solved with the BEM are presented to show the correctness of the thin plate BIEs for multi-connected domains. Finally, Section 6 concludes the paper.

2. BIE formulation for thin plate bending problems

For the sake of completeness and discussions in the following sections, we first review the governing equations and the direct BIE formulations for general thin plate bending problems. These formulations are well documented in the BEM literature [1–11].

Consider an elastic thin plate with its middle surface occupying a 2D domain V with boundary S (Fig. 1). The deflection $w(\mathbf{x})$ of the plate is governed by the following biharmonic equation [24]:

$$D\nabla^4 w(\mathbf{x}) = q(\mathbf{x}), \quad \mathbf{x} \in V \tag{1}$$

where $D = Eh^3/12(1-\nu^2)$ is the bending rigidity, E Young's modulus, ν Poisson's ratio, h the thickness, and q a distributed load in the lateral direction. The bending and twisting moments M_{ij} are

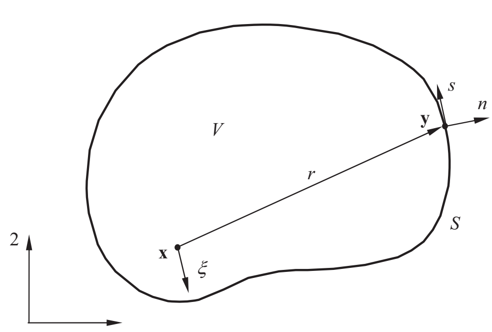

Fig. 1. A domain V with boundary S.

related to the deflection w by the following relationship (index notation is used in this paper when it is convenient):

$$M_{ij} = -D\left[\nu w_{,kk}\delta_{ij} + (1-\nu)w_{,ij}\right] \tag{2}$$

where $(\ )_{,i} = \partial(\ )/\partial x_i$ (i=1 and 2) and summation over repeated index is assumed. On boundary S, the bending and twisting moments are given by:

$$\left.\begin{aligned} M_n &= M_{ij}n_i n_j = -D\left[\nu\nabla^2 w + (1-\nu)w_{,nn}\right] \\ M_{ns} &= M_{ij}n_i s_j = -D(1-\nu)w_{,ns} \end{aligned}\right\} \tag{3}$$

respectively, where n_i and s_i are the direction cosines of the outward normal n and tangential direction s of boundary S, (Fig. 1), and no summation is assumed over n and s. On the boundary, the shear force Q_n and the Kirchhoff equivalent shear force K_n are given by:

$$\left.\begin{aligned} Q_n &= M_{ij,j}n_i = -D(\nabla^2 w)_{,n} \\ K_n &= Q_n + M_{ns,s} = -D(\nabla^2 w)_{,n} + M_{ns,s} \end{aligned}\right\} \tag{4}$$

respectively, where no summation over s is assumed in the second equation.

The governing Eq. (1) is solved under given boundary conditions to obtain the deflection w of the plate. Once w is known, the bending and twisting moments and shear forces can be determined by Eqs. (2)–(4).

Governing Eq. (1) can be transformed into a set of integral equations, based on the following fundamental solution $w^*(\mathbf{x},\mathbf{y})$ for plate bending problems, which satisfies the following governing equation:

$$D\nabla^4 w^*(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x},\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in R^2 \tag{5}$$

where $\nabla^4(\ ) = (\ )_{,iijj}$ is taken at field point $\mathbf{y}$, $\delta(\mathbf{x},\mathbf{y})$ is the Dirac δ-function representing a concentrated unit force acting at source point $\mathbf{x}$ in the lateral direction, and R^2 is the full 2D space. The expression of the fundamental solution $w^*(\mathbf{x},\mathbf{y})$ is given by [1,5–8]:

$$w^* = w^*(\mathbf{x},\mathbf{y}) = \frac{1}{8\pi D}r^2 \log r \tag{6}$$

where $r = |\mathbf{y}-\mathbf{x}|$, and $\mathbf{x}$ and $\mathbf{y}$ can be any two points in the 2D space. The solution w^* represents the deflection of an infinitely large plate at $\mathbf{y}$ due to the unit force acting at $\mathbf{x}$.

Substituting Eq. (6) into Eqs. (2)–(4) and taking derivatives at field point $\mathbf{y}$, we obtain the following expressions for the rotation (or normal slope), bending moment and Kirchhoff equivalent shear force, respectively, corresponding to the fundamental solution $w^*(\mathbf{x},\mathbf{y})$ [1,5–8]:

$$\theta^*(\mathbf{x},\mathbf{y}) = \frac{\partial w^*}{\partial n} = \frac{1}{8\pi D}(1 + 2\log r)r\cos\beta \tag{7}$$

$$M_n^*(\mathbf{x},\mathbf{y}) = -\frac{1}{8\pi}\left[2(1+\nu)(1+\log r) + (1-\nu)\cos 2\beta\right] \tag{8}$$

$$K_n^*(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi r}\left[2 + (1-\nu)\cos 2\beta\right]\cos\beta + \frac{1-\nu}{4\pi\rho}\cos 2\beta \tag{9}$$

where $\cos\beta = r_{,k}(\mathbf{y})n_k(\mathbf{y})$ with β being the angle between direction r and n, and $1/\rho$ is the curvature of the boundary curve S at point $\mathbf{y}$ (which is positive when the center of the curvature is on the opposite side of normal n).

Applying the Rayleigh-Green identity with w and w^*:

$$\begin{aligned} &\int_V \left[w\nabla^4 w^* - w^*\nabla^4 w\right] dV(\mathbf{y}) \\ &= \int_S \left[w(\nabla^2 w^*)_{,n} - w^*(\nabla^2 w)_{,n} + w^*_{,n}\nabla^2 w - w_{,n}\nabla^2 w^*\right] dS(\mathbf{y}) \end{aligned} \tag{10}$$

and the following identity [1,6]:

$$\int_S \left[w(\nabla^2 w^*)_{,n} - w^*(\nabla^2 w)_{,n} + w^*_{,n}\nabla^2 w - w_{,n}\nabla^2 w^* \right] dS(\mathbf{y})$$
$$= \frac{1}{D}\int_S \left[w^* K_n - K_n^* w + M_n^*\theta - \theta^* M_n \right] dS(\mathbf{y}) \tag{11}$$

for a smooth boundary S, one can derive the following first (deflection or singular) integral equation for thin plate bending problems [1,5–8]:

$$\int_S \left[w^* K_n - K_n^* w + M_n^*\theta - \theta^* M_n \right] dS(\mathbf{y}) + \int_V w^* q dV(\mathbf{y})$$
$$= \begin{cases} w(\mathbf{x}), & \mathbf{x}\in V; \\ \frac{1}{2}w(\mathbf{x}), & \mathbf{x}\in S \text{ (smooth)}; \\ 0, & \mathbf{x}\notin V\cup S \end{cases} \tag{12}$$

in which $\theta=\partial w/\partial n=\theta(\mathbf{y})$, $M_n=M_n(\mathbf{y})$, and $K_n=K_n(\mathbf{y})$ are the rotation, bending moment, and Kirchhoff equivalent shear force, respectively, corresponding to the deflection field $w(\mathbf{y})$. We only consider plates with smooth boundaries in this paper for simplicity. The jump (discontinuous) terms [1,5–8] at possible corners of a plate are therefore not present. These jump terms do not affect the analytical results to be derived and the numerical results to be presented in this paper.

The second (rotation or hypersingular) integral equation is given by [1,5–8]:

$$\int_S \left[w^*_{,\xi} K_n - K^*_{n,\xi} w + M^*_{n,\xi}\theta - \theta^*_{,\xi} M_n \right] dS(\mathbf{y}) + \int_V w^*_{,\xi} q dV(\mathbf{y})$$
$$= \begin{cases} w_{,\xi}(\mathbf{x}), & \mathbf{x}\in V; \\ \frac{1}{2}w_{,\xi}(\mathbf{x}), & \mathbf{x}\in S \text{ (smooth)}; \\ 0, & \mathbf{x}\notin V\cup S \end{cases} \tag{13}$$

in which ξ indicates a direction associated with the source point $\mathbf{x}$ (Fig. 1). On the boundary S, we have $\xi=n(\mathbf{x})$ and $w_{,\xi}(\mathbf{x})=\theta(\mathbf{x})$.

When $\mathbf{x}\in S$, Eqs. (12) and (13) are the BIEs which can be applied to solve for the two unknown boundary values among w, θ, M_n and K_n under given boundary conditions. Once all the boundary values are known, Eqs. (12) and (13) with $\mathbf{x}\in V$ (also called representation integrals) can be applied to evaluate the deflection and rotation of the plate inside domain V.

3. Identities for the fundamental solution

We derive four integral identities for the fundamental solution $w^*(\mathbf{x},\mathbf{y})$ using the integral Eqs. (12) and (13) first. These results are very similar to the ones for the fundamental solutions of the potential and elasticity problems [12–14].

For governing Eq. (1) with $q=0$, the expression

$$w(\mathbf{x})=c$$

where c is an arbitrary constant representing a rigid-body translation of the plate, is a solution of Eq. (1). Thus, $w(\mathbf{x})=c$ should also satisfy integral Eqs. (12) and (13) with $q=0$. Substituting $w(\mathbf{x})=c$ with $\theta=M_n=K_n=0$ into Eq. (12), we obtain the *first identity* for the fundamental solution:

$$\int_S K_n^*(\mathbf{x},\mathbf{y}) dS(\mathbf{y}) = \begin{cases} -1, & \mathbf{x}\in V; \\ -\frac{1}{2}, & \mathbf{x}\in S \text{ (smooth)}; \\ 0, & \mathbf{x}\notin V\cup S \end{cases} \tag{14}$$

The physical meaning of this identity is clear, that is, the Kirchhoff equivalent shear force on a contour S must be in equilibrium with the unit concentrated force at the source point $\mathbf{x}$ and within the domain enclosed by S.

Substituting $w(\mathbf{x})=c$ with $\theta=M_n=K_n=0$ into Eq. (13), we obtain the *second identity* for the fundamental solution:

$$\int_S K^*_{n,\xi}(\mathbf{x},\mathbf{y}) dS(\mathbf{y}) = 0, \quad \mathbf{x}\in R^2 \tag{15}$$

which can also be derived by taking the derivative of the first identity with respect to ξ at $\mathbf{x}$.

Similarly, a linear solution or a deflection due to a rigid-body rotation in the following form:

$$w(\mathbf{y}) = d_k(y_k - \bar{y}_k)$$

also satisfies governing Eq. (1) with $q=0$, where d_k is the component of an arbitrary vector and $\bar{y}_k$ are the coordinates of a reference point $\bar{\mathbf{y}}$ on the 2D plane. Therefore, substituting $w(\mathbf{y})=d_k(y_k-\bar{y}_k)$ with $\theta(\mathbf{y})=\partial w/\partial n = d_k\partial y_k/\partial n = d_k n_k(\mathbf{y})$ and $M_n=K_n=0$ into Eq. (12), we obtain the *third identity* for the fundamental solution:

$$\int_S \left[M_n^*(\mathbf{x},\mathbf{y}) n_k(\mathbf{y}) - K_n^*(\mathbf{x},\mathbf{y})(y_k-\bar{y}_k) \right] dS(\mathbf{y})$$
$$= \begin{cases} (x_k-\bar{y}_k), & \mathbf{x}\in V; \\ \frac{1}{2}(x_k-\bar{y}_k), & \mathbf{x}\in S \text{ (smooth)}; \\ 0, & \mathbf{x}\notin V\cup S \end{cases} \tag{16}$$

Similarly, substituting $w(\mathbf{y})=d_k(y_k-\bar{y}_k)$ with $\theta(\mathbf{y})=d_k n_k(\mathbf{y})$ and $M_n=K_n=0$ into Eq. (13), we obtain the *fourth identity* for the fundamental solution:

$$\int_S \left[M^*_{n,\xi}(\mathbf{x},\mathbf{y}) n_k(\mathbf{y}) - K^*_{n,\xi}(\mathbf{x},\mathbf{y})(y_k-\bar{y}_k) \right] dS(\mathbf{y})$$
$$= \begin{cases} \xi_k, & \mathbf{x}\in V; \\ \frac{1}{2}\xi_k, & \mathbf{x}\in S \text{ (smooth)}; \\ 0, & \mathbf{x}\notin V\cup S \end{cases} \tag{17}$$

where ξ_k is the direction cosine of ξ. This can also be derived by taking the derivative of the third identity with respect to ξ at $\mathbf{x}\notin S$ and go through a limit process for the result at $\mathbf{x}\in S$.

Note that in all the four identities, the normal n is the outward normal of the domain enclosed by the closed curve S. Therefore, to apply these identities, the normal must be consistent with this definition. In the results for potential and elasticity problems [12–14] corresponding to identities (16) and (17), the reference point $\bar{\mathbf{y}}$ is selected as the same as the source point $\mathbf{x}$, that is, $\bar{y}_k=x_k$ and thus simpler or results of special cases are obtained [12–14].

These four identities can also be derived by directly integrating governing Eq. (5) for the fundamental solution, as in the cases of potential and elasticity problems [12–14], without using integral Eqs. (12) and (13) and introducing the rigid-body motion (or simple) solutions. To show this, we integrate the governing Eq. (5) over a domain V to write

$$\int_V D\nabla^4 w^*(\mathbf{x},\mathbf{y}) dV(\mathbf{y}) = \int_V \delta(\mathbf{x},\mathbf{y}) dV(\mathbf{y}) \tag{18}$$

Applying the Gauss theorem and the relationship in Eq. (4), the left-hand side is evaluated as

$$\int_V D\nabla^4 w^*(\mathbf{x},\mathbf{y}) dV(\mathbf{y}) = D\int_V w^*_{,iijj} dV(\mathbf{y}) = D\int_S w^*_{,iij} n_j dS(\mathbf{y})$$
$$= D\int_S (\nabla^2 w^*)_{,n} dS(\mathbf{y}) = \int_S \left(-K_n^* + M^*_{ns,s}\right) dS(\mathbf{y}) = -\int_S K_n^* dS(\mathbf{y})$$

Substituting this result into Eq. (18) and applying the sifting property for Dirac δ-function [14,25], we obtain the first identity (14). Taking the derivative of Eq. (5) with respect to a direction ξ at $\mathbf{x}$ and integrate at $\mathbf{y}$ over domain V, we can also obtain the second identity (15).

Next, we multiply both sides of Eq. (5) by $(y_k - \bar{y}_k)$ and integrate over domain V to write

$$\int_V (y_k - \bar{y}_k) D\nabla^4 w^*(\mathbf{x},\mathbf{y}) dV(\mathbf{y}) = \int_V (y_k - \bar{y}_k)\delta(\mathbf{x},\mathbf{y}) dV(\mathbf{y}) \tag{19}$$

The left-hand side is evaluated as

$$\int_V (y_k - \bar{y}_k) D\nabla^4 w^*(\mathbf{x},\mathbf{y}) dV(\mathbf{y}) = D\int_V \left[(y_k - \bar{y}_k) w^*{,_{iij}}\right]_{,j} dV(\mathbf{y}) - D\int_V w^*{,_{iik}} dV(\mathbf{y})$$

$$= D\int_S \left[(y_k - \bar{y}_k) w^*{,_{iij}}\right] n_j dS\big(\mathbf{y}\big) - D\int_S w^*{,_{ii}} n_k dS(\mathbf{y})$$

$$= D\int_S \left[\Big(y_k - \bar{y}_k\Big)\Big(\nabla^2 w^*\Big)_{,n} - n_k \nabla^2 w^*\right] dS(\mathbf{y})$$

Applying the results in Eq. (11) with $w = y_k - \bar{y}_k$, $w_{,n} = \theta = n_k$, $\nabla^2 w = M_n = K_n = 0$, the above expression becomes

$$\int_V (y_k - \bar{y}_k) D\nabla^4 w^*(\mathbf{x},\mathbf{y}) dV(\mathbf{y}) = \int_S \left[M_n^* n_k - K_n^*(y_k - \bar{y}_k)\right] dS(\mathbf{y})$$

Substituting this result into Eq. (19) and applying the sifting property for Dirac δ-function [14,25], we obtain the third identity (16). Taking the derivative of Eq. (5) with respect to a direction ξ at $\mathbf{x}$, multiplying the result by $(y_k - \bar{y}_k)$ and integrating over domain V, we can also derive the fourth identity (17).

The four integral identities in Eqs. (14)–(17) can be applied to derive the weakly-singular and nonsingular forms of BIEs (12) and (13) with $\mathbf{x} \in S$ for plate bending problems, as in the cases of the BIEs for potential and elasticity problems [12–14]. They can also be employed to show the nonuniqueness of the solutions of BIEs for multi-connected domains.

4. Nonuniqueness of the solution with the hypersingular BIE for multi-connected domains

We now investigate the solutions of singular BIE (12) and hypersingular BIE (13) for multi-connected domains (Fig. 2) to see if the solutions are unique or not on the edge of a hole. To be precise, by nonuniqueness we mean here that the HBIE can admit an arbitrary constant on the edge of a hole in the solution of a multi-connected domain plate bending problem. This solution can be obtained by other means, for example, by using the BEM, FEM or analytical methods.

In Fig. 2, S_o is the outer boundary and S_i is a typical internal boundary which is not constrained (for example, the free edge of a hole in a plate).

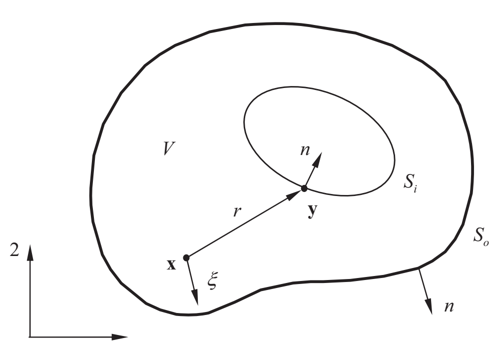

Fig. 2. A multi-connected domain V with outer boundary So and inner boundary S_i ($S_o \cup S_i = S$).

Assume $w(\mathbf{y})$ is a solution to BIE (12) or BIE (13). We consider the following function:

$$\tilde{w}(\mathbf{y}) = \begin{cases} w(\mathbf{y}), & \mathbf{y} \in S_o; \\ w(\mathbf{y}) + \left[c + d_k(y_k - \bar{y}_k)\right], & \mathbf{y} \in S_i \end{cases} \tag{20}$$

obtained by adding a rigid-body translation c and rotation $d_k(y_k - \bar{y}_k)$ to $w(\mathbf{y})$ on S_i. The rotation represented by $d_k(y_k - \bar{y}_k)$ is about a line that passes through the point $(\bar{y}_1, \bar{y}_2)$ and is normal to the vector d_k. We assume that this rigid-body solution also extends into the domain so that the corresponding normal derivative is calculated as $d_k n_k(\mathbf{y})$.

We first investigate if the expression for $\tilde{w}(\mathbf{y})$ in Eq. (20) is also a solution of BIE (12). To do this, substituting $\tilde{w}(\mathbf{y})$ into BIE (12) with $\mathbf{x} \in S_o$ first, we have

$$\int_{S_o \cup S_i} \left[w^* K_n - K_n^* w + M_n^* \theta - \theta^* M_n\right] dS(\mathbf{y}) + \int_V w^* q dV(\mathbf{y})$$
$$+ \int_{S_i} \left[-K_n^*(c + d_k(y_k - \bar{y}_k)) + M_n^*(d_k n_k(\mathbf{y}))\right] dS(\mathbf{y}) = \frac{1}{2} w(\mathbf{x}), \quad \mathbf{x} \in S_o$$

which is reduced to

$$-c\int_{S_i} K_n^* dS(\mathbf{y}) + d_k \int_{S_i} \left[M_n^* n_k(\mathbf{y}) - K_n^*(y_k - \bar{y}_k)\right] dS(\mathbf{y}) = 0, \quad \mathbf{x} \in S_o \tag{21}$$

by applying BIE (12) for w. The above equation is satisfied by using first identity (14) and third identity (16) applied to the domain enclosed by S_i with $\mathbf{x}$ outside this domain. Notice that when the direction of the normal is reversed (Fig. 3(b), here, for simplicity, the same symbol n is used for the reversed normal), which is required before we can apply the identities, K_n^* given by Eq. (9) changes the sign, while M_n^* given by Eq. (8) does not.

Similarly, substituting $\tilde{w}(\mathbf{y})$ into BIE (12) with $\mathbf{x} \in S_i$, we have

$$-c\int_{S_i} K_n^* dS(\mathbf{y}) + d_k \int_{S_i} \left[M_n^* n_k(\mathbf{y}) - K_n^*(y_k - \bar{y}_k)\right] dS(\mathbf{y})$$
$$= \frac{1}{2}\left[c + d_k(x_k - \bar{y}_k)\right], \quad \mathbf{x} \in S_i$$

by applying BIE (12) for w. After reversing the direction of the normal (Fig. 3(b)), this equation is changed to

$$c\left[\int_{S_i} K_n^* dS(\mathbf{y}) - \frac{1}{2}\right] - d_k \left\{\int_{S_i} \left[M_n^* n_k(\mathbf{y}) - K_n^*(y_k - \bar{y}_k)\right] dS(\mathbf{y}) + \frac{1}{2}(x_k - \bar{y}_k)\right\} = 0, \quad \mathbf{x} \in S_i$$

Using first identity (14) and third identity (16), we have from the above result

$$c\left[-\frac{1}{2} - \frac{1}{2}\right] - d_k\left[\frac{1}{2}(x_k - \bar{y}_k) + \frac{1}{2}(x_k - \bar{y}_k)\right] = 0, \quad \mathbf{x} \in S_i$$

or

$$c + d_k(x_k - \bar{y}_k) = 0, \quad \mathbf{x} \in S_i \tag{22}$$

This cannot be satisfied with arbitrary values of c, d_k and x_k, unless $c = 0$ and $d_k = 0$. This shows that the singular or deflection BIE solution *does not* admit any rigid-body translation or rotation term on the edge of a hole. Therefore, the singular BIE solution is unique on these edges in multi-connected domains.

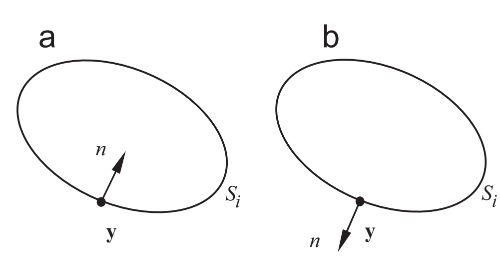

Fig. 3. Switching the direction of the normal on S_i before applying the identities.

Next, substituting $\tilde{w}(\mathbf{y})$ into BIE (13) with $\mathbf{x} \in S_o$, we have

$$-c\int_{S_i} K^*_{n},_{\xi} dS\left(\mathbf{y}\right) + d_k \int_{S_i} \left[M^*_{n},_{\xi} n_k(\mathbf{y}) - K^*_{n},_{\xi}(y_k - \bar{y}_k)\right] dS(\mathbf{y}) = 0, \quad \mathbf{x} \in S_o \tag{23}$$

by applying BIE (13) for w, which is satisfied using second identity (15) and fourth identity (17) after reversing the direction of normal n on S_i.

Similarly, substituting $\tilde{w}(\mathbf{y})$ into BIE (13) with $\mathbf{x} \in S_i$, we have

$$-c\int_{S_i} K^*_{n},_{\xi} dS(\mathbf{y}) + d_k \int_{S_i} \left[M^*_{n},_{\xi} n_k(\mathbf{y}) - K^*_{n},_{\xi}(y_k - \bar{y}_k)\right] dS(\mathbf{y}) = \frac{1}{2} d_k \xi_k, \quad \mathbf{x} \in S_i$$

by applying BIE (13) for w. This equation is reduced to

$$d_k \left\{ \int_{S_i} \left[M^*_{n},_{\xi} n_k(\mathbf{y}) - K^*_{n},_{\xi}(y_k - \bar{y}_k)\right] dS(\mathbf{y}) + \frac{1}{2}\xi_k \right\} = 0, \quad \mathbf{x} \in S_i$$

by applying second identity (15) and reversing the directions of normals n and ξ on S_i (Fig. 3(b)). Note that the sign of $M^*_{n},_{\xi}$ changes, while the sign of $K^*_{n},_{\xi}$ does not, when the two normals are switched. From fourth identity (17) the above expression is further reduced to

$$d_k \left\{ \frac{1}{2}\xi_k + \frac{1}{2}\xi_k \right\} = 0, \quad \mathbf{x} \in S_i$$

or

$$d_k \xi_k = 0, \quad \mathbf{x} \in S_i \tag{24}$$

Eq. (24) cannot hold because ξ_k changes its value on S_i. Therefore, we must conclude that $d_k = 0$. That is, the hypersingular BIE (13) solution can contain an arbitrary rigid-body translation c on the edge of a hole, but cannot contain any rigid-body rotation term. We conclude that the solution of BIE (13) is *not unique* on the edges of holes in multi-connected domains.

The above observations and proofs have not been reported in the literature for the BIEs related to the plate bending problems. This defect related to the hypersingular or traction BIEs for multi-connected domains has been reported in Refs. [15,16] for elasticity problems and in Refs. [17,18] for Stokes flow problems. In Ref. [15], an integral representation for the domain enclosed by the hole is applied to determine the unknown constant displacement in

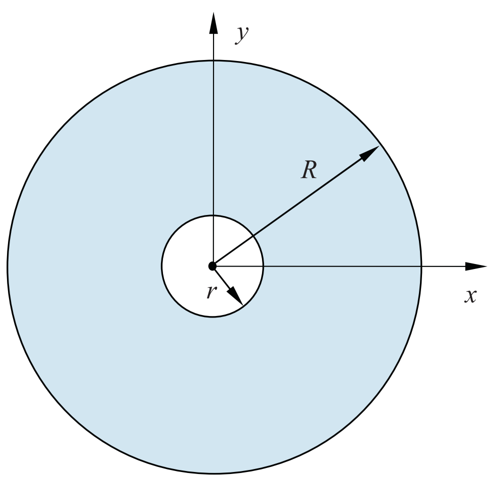

Fig. 4. A circular plate with a center hole.

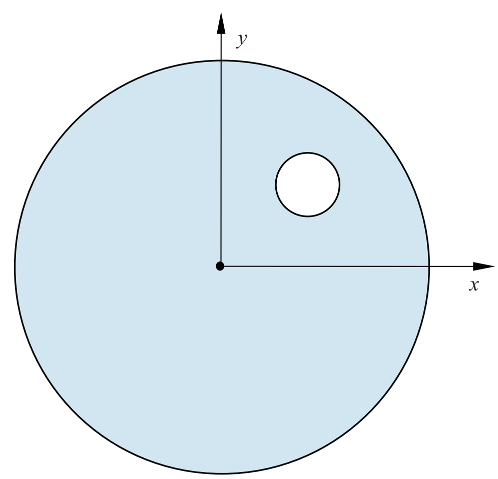

Fig. 6. A circular plate with an off-center hole.

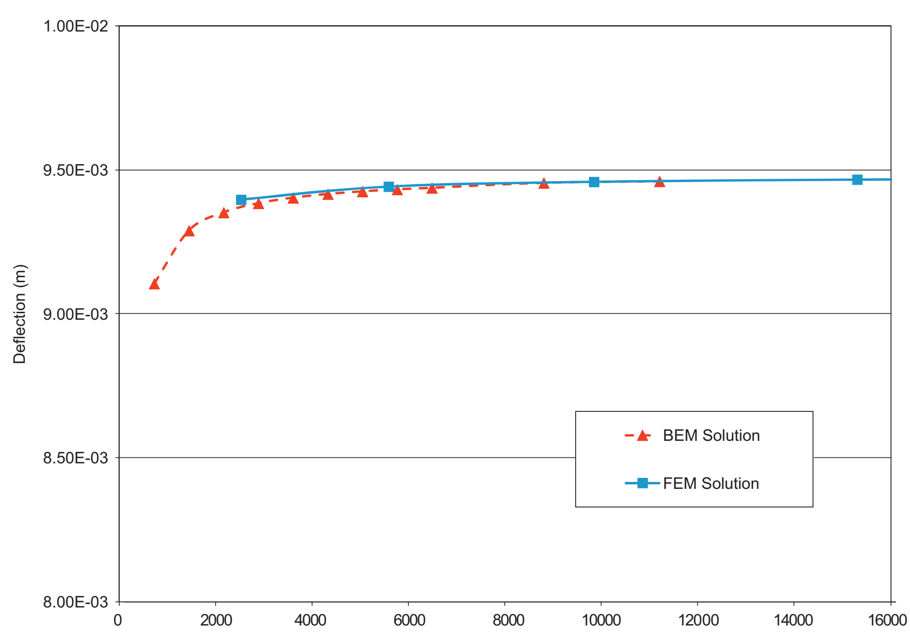

Fig. 5. Computed maximum deflection for the circular plate with a center hole.

the solution on edge of the hole for the original traction BIE for the multi-domain problems. A few selected constraints on the edge of a hole are also introduced in order to remove the rigid-body motion in the solution of the traction BIE. In Ref. [16], it is proposed to apply the displacement BIE in the Galerkin BEM, instead of the traction BIE, on the traction-free edge of a hole, in order to avoid the nonuniqueness of the displacement solution of the traction BIE. For Stokes flow problems, a dual BIE formulation using a linear combination of the singular and hypersingular BIEs is suggested in Refs. [17,18].

However, a remedy to the above nonuniqueness problem has already been used in the BIE formulation for plate bending problems, since both the singular (deflection) BIE (12) and the hypersingular (rotation) BIE (13) are required in solving plate bending problems using the BIEs. The plate bending BIE formulation is another type of dual BIE formulations [17,18] that again demonstrate their robustness and benefits in the BEM, even they are sometimes unnoticed.

5. Numerical examples

Although the theory has confirmed that the solution of the dual BIE for plate bending problems yields unique solutions for plates with holes when modeled with the BEM, it is necessary to support this assertion with a few examples. To the authors' best knowledge, there are very few examples of plates with holes reported in the BEM literature. A square plate with a square hole was solved with the BEM first by Hartmann ([26], p. 282) and later by Aliabadi ([11], p. 151). The BEM results for the deflection and bending moment along selected cross sections inside the domain were compared with the FEM results. Reasonable agreements between the BEM and FEM results were obtained. Similar results were obtained by Paris and Leon using a circular plate with a circular hole [27]. However, in all these studies, the BEM deflection solutions on edges of the holes were not reported. Whether or not the BEM solutions on the edges contain arbitrary rigid-body translation displacements is not known, since these rigid-body solutions on edges do not affect the results inside the domain. Therefore, concerns remain with the BIE solutions for plates with holes. Further examples are necessary.

Four examples are given in this section to show the correctness of the BIE formulation for solving plate bending problems using plates with holes. In all the examples, Young's modulus $E = 70$ GPa, Poisson's ratio $\nu = 0.3$, thickness of the plate $h = 0.05$ m, and the same uniform distributed load $q = 0.1$ MPa is applied. The domain integrals in BIEs (12) and (13) involving the distributed load q are converted into boundary integrals before the BEM discretizations. Constant line elements are applied in the BEM, where all the singular and hypersingular integrals are calculated analytically based on the definitions of the CPV and HFP integrals [18], respectively. In each case, the BEM solution

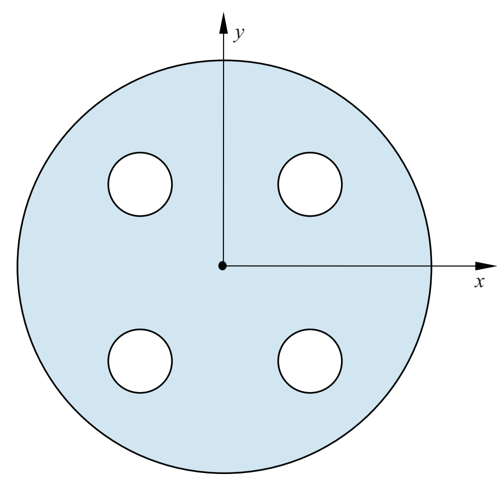

Fig. 8. A circular plate with four holes.

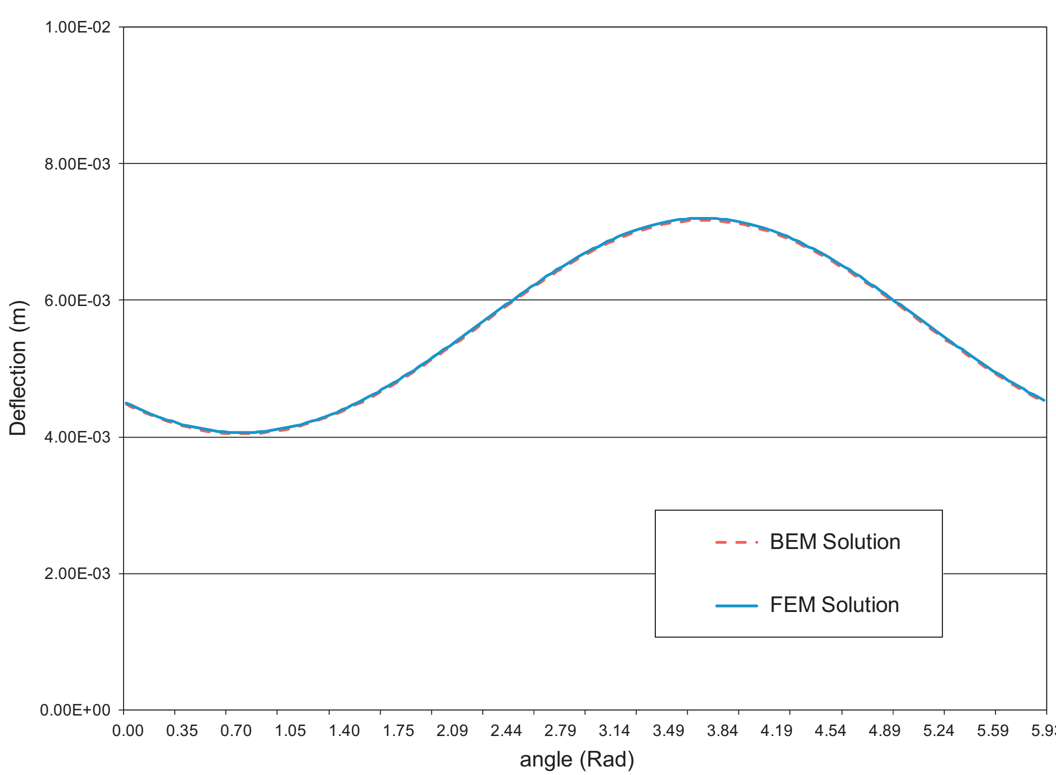

Fig. 7. Deflection on the edge of the hole for the circular plate with an off-center hole.

of the deflection on the edge of a hole is reported and comparison with the FEM solution is shown.

5.1. *A circular plate with a center hole*

We first consider a circular plate of radius $R=1$ m with a center hole of radius $r=0.25$ m (Fig. 4) to see if there is any constant value in the deflection solution on the edge of a hole using the BIEs. The outer boundary of the plate is simply supported and the inner boundary (edge of the hole) is free. Under the uniform distributed load q, the maximum deflection is on the edge of the hole. The BEM results are obtained with several meshes and compared with the FEM (ANSYS®) solutions using 4-node shell elements. A comparison of the convergence of the BEM and FEM solutions is shown in Fig. 5, and good agreement is observed. Both the BEM and FEM solutions converged as the numbers of the elements are sufficiently large. This suggests that no constant value presents in the deflection solution on the edge of the hole using the BEM.

5.2. *A circular plate with an off-center hole*

Next we consider the circular plate with an off-center hole as shown in Fig. 6 to further investigate if there are any constant translation and/or rotation terms in the BEM solution of the edge of a hole. The radius of the plate is $R=1$ m, the radius of the hole is $r=0.15$ m, and the hole is centered at $x=y=0.4$ m. Again the plate is simply supported on the outer boundary and the edge of the hole is free. Fig. 7 shows the computed values of the deflection on the edge of the hole using the BEM and compared with those using the FEM. A total of 3240 line elements are used in the BEM model and a total of 12,610 shell elements are used in the FEM model for this comparison study. Very good agreement between the BEM results and FEM results is achieved as shown in Fig. 7, which further suggests that the BEM solution is uniquely determined on the edge of the hole.

5.3. *A circular plate with four holes*

A circular plate with four circular holes are considered next. The plate has the same dimensions as in the previous case, except for the addition of three more holes in the second, third and fourth quadrants (Fig. 8). The plate is simply supported on the outer boundary, while the edges of all four holes are free. As in the previous case, the values of the computed deflection on the edge of the hole in the first quadrant using the BEM and FEM are plotted in Fig. 9. For the BEM model 5960 line elements are used, and for the FEM model 10,836 shell elements are used. In spite of the increased complexity of the geometry of the plate, the agreement of the BEM and FEM results are still very good as shown in Fig. 9.

5.4. *A square plate with four holes*

Finally, a square plate with four circular holes (Fig. 10) is analyzed using the BEM and the results are compared with those

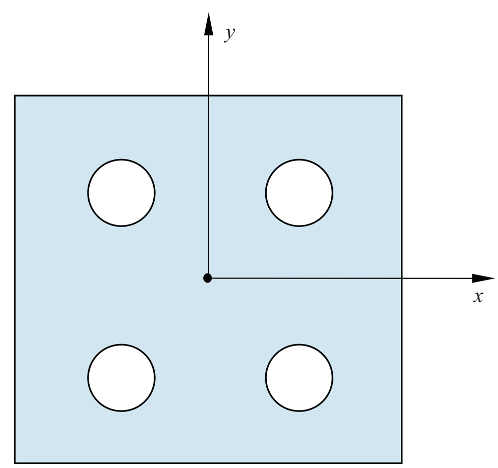

Fig. 10. A square plate with four holes.

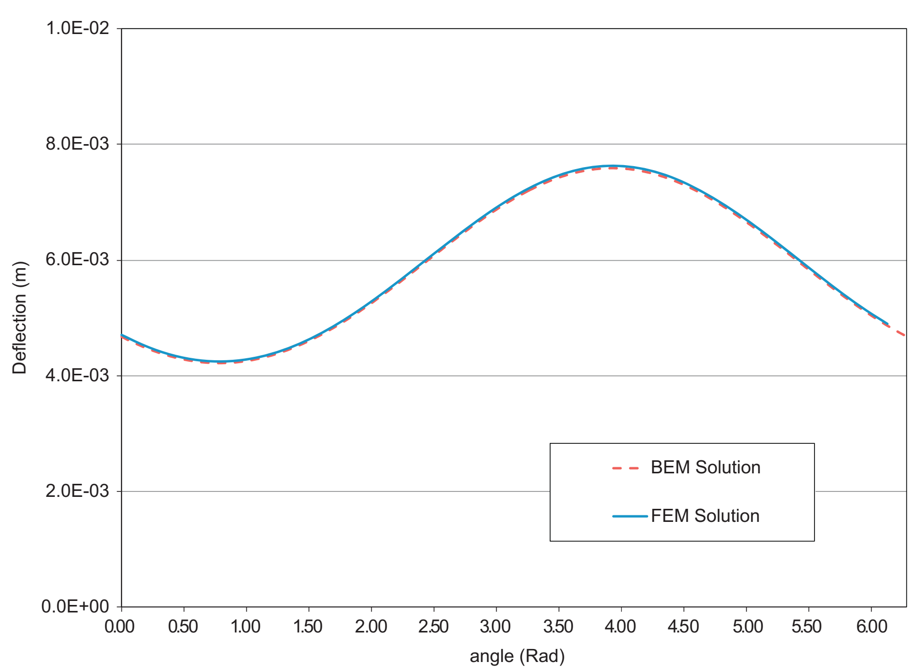

Fig. 9. Deflection on the edge of a hole for the circular plate with four holes.

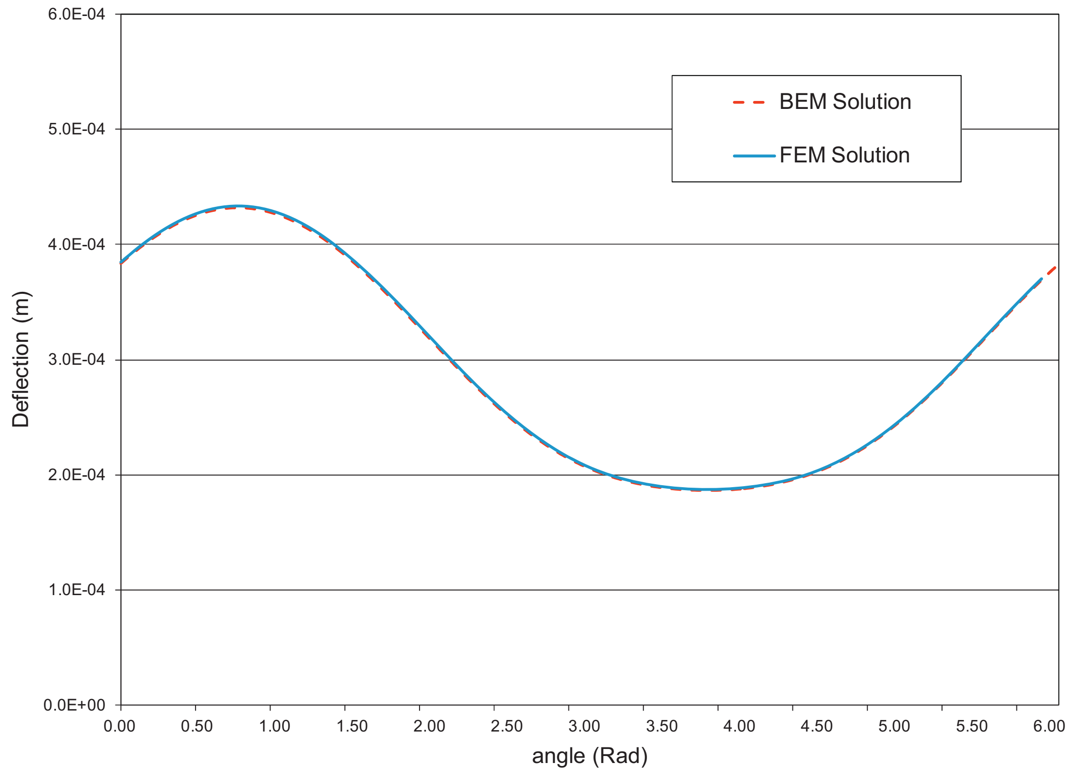

Fig. 11. Deflection on the edge of a hole for the square plate with four holes.

using the FEM. The square plate has an edge length of 1 m, the four holes have the same radius of 0.1 m and are centered at $x=y=\pm 0.25$ m. The outer boundary of the square plate is simply supported while the edges of the four holes are free. The same uniform distributed load q is applied to the plate. The BEM model has 5600 line elements, while the FEM uses 12,480 shell elements. Excellent agreement between the BEM and FEM results are obtained as shown in Fig. 11, in which the values of the computed deflection on the edge of the hole in the third quadrant are plotted. Note that the BIEs we used do not have the corner jump terms, which when included in the BIEs can further improve the accuracy of the BEM solutions for domains with corners. One argument for ignoring the corner terms is that one can always consider that any sharp corners can be rounded off with curved boundaries.

The above four examples show that the dual BIE formulation for plate bending problems can provide unique solutions for plates with multi-connected domains and excellent BEM results can be obtained even with constant boundary elements. However, as the number of holes increases (such as with a perforated plate), the complexity of the geometry will present difficulties to both the conventional BEM used here and the FEM as more and more elements will be required in the analysis. The fast multipole BEM will need to be employed in such cases as for the 2D elasticity analysis of perforated plates [18,28]. The fast multipole BEM for large-scale modeling of plate bending problems is being investigated and will be reported in a follow-up paper.

6. Conclusion

Four integral identities for the fundamental solution of thin plate bending problems are derived by using the rigid-body solutions in the BIEs and by direct integration of the governing equation for the fundamental solution. These identities can be applied to derive weakly-singular and nonsingular forms of the BIEs for plate bending problems, as has been done earlier with the BIEs for potential and elasticity problems. These identities can also be applied to show that the singular BIE solution is unique while the hypersingular BIE solution is not unique on the edges of holes in plates on multi-connected domains, which is reported for the first time in this paper. It is proved that the solution of the hypersingular BIE can have an arbitrary rigid-body translation term on the edge of a hole, and therefore deflection solution is not unique on the edge of the hole using the hypersingular BIE alone. However, since both the singular and hypersingular BIEs are required in solving a plate bending problem using the BEM, the BEM solution is always uniquely determined on edges of holes for multi-connected domains. The four examples of plates with holes and solved with the BEM demonstrate the correctness and effectiveness of the BEM based on the dual BIE formulation for solving multi-connected domain problems.

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References

[1] Stern M. A general boundary integral formulation for the numerical solution of plate bending problems. Int J Solids Struct 1979;15:769–82.
[2] Morjaria M, Mukherjee S. Inelastic analysis of transverse deflection of plates by the boundary element method. J Appl Mech 1980;47:291.
[3] Tanaka M. Integral equation approach to small and large displacements of thin elastic plate. In: Brebbia CA, (Ed..) Proceedings of the fourth

international seminar on boundary element methods in engineering. Southampton, England: Springer-Verlag; 1982. p. 526.
[4] Kamiya N, Sawaki Y. An integral equation approach to finite deflection of elastic plates. Int J Nonlinear Mech 1982;17:187–94.
[5] Stern M. Boundary integral equations for bending of thin plates. In: Brebbia CA, editor. Progress in boundary elements. London: Pentech Press; 1983.
[6] Liu YJ. Finite deflection analysis of thin elastic plate by the boundary integral equation method. MS thesis. Aerospace Engineering Department, Northwestern Polytechnical University; 1984 [in Chinese].
[7] Ye TQ, Liu YJ. Finite deflection analysis of elastic plate by the boundary element method. Appl Math Modelling 1985;9:183–8.
[8] Hartmann F, Zotemantel R. The direct boundary element method in plate bending. Int J Numer Methods Eng 1986;23:2049–69.
[9] Liu YJ. Elastic stability analysis of thin plate by the boundary element method — a new formulation. Eng Anal Boundary Elem 1987;4:160–4.
[10] Beskos DE, editor. Boundary element analysis of plates and shells. Berlin: Springer-Verlag; 1991.
[11] Aliabadi MH. The boundary element method — applications in solids and structures, vol. 2. Chichester: Wiley; 2002.
[12] Liu YJ, Rudolphi TJ. Some identities for fundamental solutions and their applications to weakly-singular boundary element formulations. Eng Anal Boundary Elem 1991;8:301–11.
[13] Liu YJ, Rudolphi TJ. New identities for fundamental solutions and their applications to non-singular boundary element formulations. Comput Mech 1999;24:286–92.
[14] Liu YJ. On the simple-solution method and non-singular nature of the BIE/BEM — a review and some new results. Eng Anal Boundary Elem 2000;24:787–93.
[15] Frangi A, Novati G. Symmetric BE method in two-dimensional elasticity: evaluation of double integrals for curved elements. Comput Mech 1996;19:58–68.
[16] Perez-Gavilan JJ, Aliabadi MH. Symmetric Galerkin BEM for multi-connected bodies. Commun Numer Methods Eng 2001;17:761–70.
[17] Liu YJ. A new fast multipole boundary element method for solving 2-D stokes flow problems based on a dual BIE formulation. Eng Anal Boundary Elem 2008;32:139–51.
[18] Liu YJ. Fast multipole boundary element method — theory and applications in engineering. Cambridge: Cambridge University Press; 2009.
[19] Chen JT, Wu C, Chen K, Lee Y. Degenerate scale for the analysis of circular thin plate using the boundary integral equation method and boundary element methods. Comput Mech 2006;38:33–49.
[20] Chen JT, Lin JH, Kuo SR, Chyuan SW. Boundary element analysis for the Helmholtz eigenvalue problems with a multiply connected domain. Proc R Soc London Ser A: Math, Phys Eng Sci 2001;457:2521–46.
[21] Chen JT, Liu LW, Hong H-K. Spurious and true eigensolutions of Helmholtz BIEs and BEMs for a multiply connected problem. Proc R Soc London Ser A: Math, Phys Eng Sci 2003;459:1891–924.
[22] Chen JT, Shen W-C. Degenerate scale for multiply connected Laplace problems. Mech Res Commun 2007;34:69–77.
[23] Chen YZ, Lin XY, Wang ZX. Evaluation of the degenerate scale for BIE in plane elasticity and antiplane elasticity by using conformal mapping. Eng Anal Boundary Elem 2009;33:147–58.
[24] Timoshenko SP, Woinowsky-Krieger S. Theory of plates and shells. 2nd ed. New York: McGraw-Hill; 1987.
[25] Zemanian AH. Distribution theory and transform analysis - an introduction to generalized functions, with applications. New York: Dover; 1987.
[26] Hartmann F. Introduction to boundary elements: theory and applications. Berlin: Springer-Verlag; 1989.
[27] Paris F, Leon SD. Boundary element method applied to the analysis of thin plates. Comput Struct 1987;25:225–33.
[28] Liu YJ. A new fast multipole boundary element method for solving large-scale two-dimensional elastostatic problems. Int J Numer Methods Eng 2006;65:863–81.