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New identities for fundamental solutions and their applications to non-singular boundary element formulations

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Abstract Based on a general, operational approach, two new integral identities for the fundamental solutions of the potential and elastostatic problems are established in this paper. Non-singular forms of the conventional boundary integral equations (BIEs) are derived by employing these two identities for the fundamental solutions and the two-term subtraction technique. Both the strongly- (Cauchy type) and weakly-singular integrals existing in the conventional BIEs are removed from the BIE formulations. The existence of the non-singular forms of the conventional BIEs raises new and interesting questions about the smoothness requirement in the boundary element method (BEM), since the two-term subtraction requires, theoretically, C^1 continuity of the density function, rather than the C^0 continuity as required by the original singular or weakly-singular forms of the conventional BIEs. Implication of the non-singular BIEs on the smoothness requirement will be discussed in this paper.

1 Introduction

Dealing with singular integrals, including the strongly-singular (Cauchy principal value type) and hypersingular (Hadamard finite part type) integrals, in the boundary integral equation (BIE) formulations has been a seemingly daunting task since the early days of the boundary element method (BEM) (see (Jaswon 1963; Rizzo 1967)). A great deal of research effort has been devoted to the subject of how to accurately compute the singular integrals analytically or numerically. However, careful studies of the BIE formulations have revealed that the BIEs can be recast in weakly-singular forms which eliminate the calculations of the singular integrals. This weakly-singular form of the conventional BIE for elastostatic problems can be obtained by using an integral expression for the coefficient matrix (C_{ij}) of the free term in the BIE formulation (see, (Cruse 1974; Rizzo and Shippy 1977)). The process leading to the weakly-singular form of the BIE demonstrates that the strongly-singular integrals in the BIE can be cancelled out completely if they have been identified. The utilization of the fundamental solutions, which are the origin of the singularity, in the BIEs should not necessarily give rise to singular formulations for the physical problems which, in most cases, are not singular at all in the first place.

The weakly-singular nature of the BIEs is quite general, not just limited to the formulation of elastostatic problems. Rudolphi (Rudolphi 1991) generalized the simple solution approach to obtain some identities for the fundamental solutions and have applied them in the derivations of the weakly-singular forms of the conventional and hypersingular BIEs. In this approach, simple solutions or modes of the problem (such as the rigid-body translation and rotation in the elastostatic case) are imposed on the BIEs and result in certain identities for the fundamental solutions. Two identities were developed and used in the regularization of the hypersingular BIE for potential problems in Ref. (Rudolphi 1991). A more general way to establish the identities for the fundamental solutions was developed in Ref. (Liu and Rudolphi 1991) based on an operational approach. In this approach the governing equations for the fundamental solutions are integrated over an arbitrary closed domain, and Gauss' theorem and the properties of the Dirac-delta function are employed to transform the domain integrals to boundary ones. This approach, although involving more mathematics, does not depend on the availability of the BIE formulations of the problems, offers more physical insights to the identities for the fundamental solutions (equilibrium of the forces, moments, etc.), and is applicable to both finite and infinite domains. The weakly-singular forms of the conventional and hypersingular BIEs for potential and elastostatic problems, in both finite and infinite, 2-D or 3-D domains, were readily established in Ref. (Liu and Rudolphi 1991). The use of the identities in deriving the weakly-singular forms of the BIEs, especially those of the hypersingular BIEs (Krishnasamy, Rizzo et al. 1991), have been applied successfully to stress analysis (Muci-Kuchler and Rudolphi 1993), acoustics (Liu and Rizzo 1992; Liu and Chen 1999), elastodynamics (Liu and Rizzo 1993) and electromagnetics (Chao, Liu et al. 1995). This approach has also been adopted in others' work as well (see, e.g., (Tanaka, Sladek et al. 1994; Johnston 1997)). This simple solution approach is successfully applied in Ref. (Poon, Mukherjee et al. 1998) to nonlinear problems for regularizing the hyper-

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singular BIE for elastoplasticity. Most recently, this approach is developed in Ref. (Mukherjee, Shah et al. 1999) for regularizing the hypersingular BIE for thermoelastic fracture mechanics problems.

An intriguing and theoretical question is: What is the implication or impact of the weakly-singular forms of the BIEs on the smoothness requirement imposed to the original BIEs? It has been shown (Krishnasamy, Rizzo et al. 1992; Martin and Rizzo 1996) and gradually accepted in the BEM community (see, e.g., (Tanaka, Sladek et al. 1994)) that for the strongly-singular or hypersingular integrals to exist in the limit as the source point goes to the boundary, the density function or its derivatives, respectively, must be Hölder continuous (at least in the neighborhood of the source point). This means that theoretically only boundary element implementations that ensure the C^0, or C^1, continuity near each collocation point can be applied in the discretizations of the conventional, or hypersingular, BIEs, respectively. In the hypersingular BIE case, this stringent (C^1 continuity) requirement has seriously hindered the applications of the hypersingular BIEs. Two questions are then obvious: (1) Do the various weakly-singular forms of the BIEs change the smoothness requirement? (2) If the answer to the previous question is "no", then can we relax the smoothness requirement in the implementation of the various weakly-singular forms of the BIEs?

Relaxation of this continuity requirement on hypersingular BIEs has been attempted by several authors using the conforming (C^0) elements for different problems (see, e.g., (Chien, Rajiyah et al. 1990; Wu, Seybert et al. 1991; Liu and Rizzo 1992; Cruse and Suwito 1993; Huang and Cruse 1994; Cruse and Richardson 1996). The validation of this relaxation received renewed attention recently (Richardson, Cruse et al. 1997; Liu and Chen 1999; Martin, Rizzo et al. 1998) due to a strong desire to do so in the BEM community.

The existence of the non-singular forms of the conventional BIEs, developed in this paper, raises new and interesting questions about the smoothness requirement in the BEM, since the two-term subtractions require, theoretically, C^1 continuity of the density functions, rather than the C^0 continuity as required by the original singular or weakly-singular forms of the conventional BIEs. A feasible remedy to avoid this dilemma is to adopt the relaxation strategy (piecewise smoothness) in the discretizations of the developed non-singular BIEs. This relaxation in the piecewise sense may clear some of the confusions in the smoothness issue and help broaden the applications of the various weakly-singular or non-singular forms of the BIEs, including the hypersingular BIEs. However, the convergence study, especially a theoretical one, or a counter-example showing divergence, for the relaxation strategy is still urgently needed before the smoothness issue in the BEM can be finally settled.

The remaining part of this paper consists of three major sections. In Sect. 2, the three integral identities for the fundamental solutions developed in Ref. (Liu and Rudolphi 1991) for potential and elastostatic problems will be reviewed. Then, one new integral identity for each of the potential and elastostatic problems will be established based on the operational approach. In Sect. 3, non-singular forms of the conventional BIEs will be derived by employing these two new integral identities. In Sect. 4, the implication of these non-singular forms of the BIEs, which require theoretically a higher order (C^1) continuity of the density function, to the smoothness requirement in the BEM will be discussed.

2 Two new integral identities for the fundamental solutions

Consider a closed domain V in the infinite space R^m, with $m = 2$ or 3 for two- or three-dimensional space, respectively, Fig. 1. In Ref. (Liu and Rudolphi 1991) (see, also (Rudolphi 1991)), we have established the following three integral identities for the fundamental solution $G(P, P_o)$ of the potential problem (index notation is used in this paper):

The first identity

$$\int_S \frac{\partial G(P,P_o)}{\partial n} \mathrm{d}S(P) = \begin{cases} -1, & \forall P_o \in V, \\ 0, & \forall P_o \in E \end{cases} . \tag{1}$$

The second identity

$$\int_S \frac{\partial^2 G(P,P_o)}{\partial n \partial n_o} \mathrm{d}S(P) = 0, \quad \forall P_o \in V \cup E . \tag{2}$$

The third identity

$$\int_S \frac{\partial G(P,P_o)}{\partial n_o} n_k(P) \mathrm{d}S(P) - \int_S \frac{\partial^2 G(P,P_o)}{\partial n \partial n_o} (x_k - x_{ok}) \mathrm{d}S(P) = \begin{cases} n_{ok}(P_o), & \forall P_o \in V, \\ 0, & \forall P_o \in E; \end{cases} \tag{3}$$

where $S = \partial V$, $E = R^m - (V \cup S)$, n_k the directional cosines of the normal n of S at the field point $P(x_k)$, n_{ok} the direction cosines of a unit vector n_o at the source point $P_o(x_{ok})$, Fig. 1. These integral identities are derived by integrating the following governing equation (and its derivative) for the fundamental solution $G(P, P_o)$:

$$\nabla^2 G(P,P_o) + \delta(P,P_o) = 0, \quad \forall P, P_o \in R^m , \tag{4}$$

over the domain V and with the source point P_o either inside or outside V, where $\delta(P, P_o)$ is the Dirac-delta function.

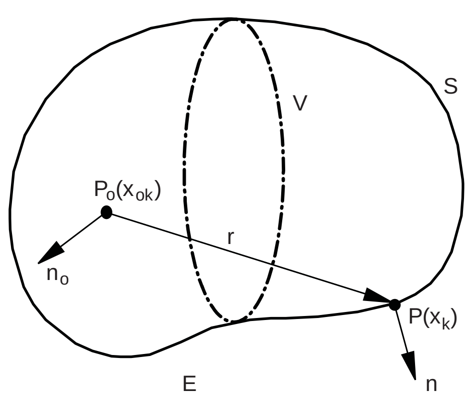

Fig. 1. A 3-D closed domain V in R^3 with boundary S (exterior domain $E = R^3 - (V \cup S)$

Similarly, the following three integral identities for the fundamental solution $U_{ij}(P,P_o)$ (Kelvin solution) of the elastostatic problem have also been established in (Liu and Rudolphi 1991):

The first identity

$$\int_S T_{ij}(P,P_o)\mathrm{d}S(P) = \begin{cases} -\delta_{ij}, & \forall P_o \in V, \\ 0, & \forall P_o \in E \ . \end{cases} \tag{5}$$

The second identity

$$\int_S \frac{\partial T_{ij}(P,P_o)}{\partial x_{ok}}\mathrm{d}S(P) = 0, \quad \forall P_o \in V \cup E \ . \tag{6}$$

The third identity

$$E_{jlpq}\int_S \frac{\partial U_{iq}(P,P_o)}{\partial x_{ok}} n_p(P)\mathrm{d}S(P) - \int_S \frac{\partial T_{ij}(P,P_o)}{\partial x_{ok}}(x_l - x_{ol})\mathrm{d}S(P) = \begin{cases} \delta_{ij}\delta_{kl}, & \forall P_o \in V, \\ 0, & \forall P_o \in E \ ; \end{cases} \tag{7}$$

where $T_{ij}(P,P_o)$ is the traction kernel corresponding to the displacement kernel $U_{ij}(P,P_o)$, E_{ijkl} the elastic modulus tensor and δ_{ij} the Kronecker delta. These three identities for the elastostatic problem are derived by integrating the following governing equations (and their derivatives) for the fundamental solution $U_{ij}(P,P_o)$:

$$\Sigma_{ijk,k}(P,P_o) + \delta_{ij}\delta(P,P_o) = 0, \quad \forall P,\ P_o \in R^m \ , \tag{8}$$

where $\Sigma_{ijk}(P,P_o) = E_{jkpq}U_{ip,q}(P,P_o)$ is the stress tensor in the fundamental solution.

Although the integral identities in (1)–(3) for the potential problem and in (5)–(7) for the elastostatic problem can also be derived by imposing certain type of simple solutions, such as rigid-body translations, to the corresponding boundary integral representations (Cruse 1974; Rizzo and Shippy 1977; Rudolphi 1991), the operational approach developed in (Liu and Rudolphi 1991) and based on the original governing equations seems to be more general and can offer more insights into the BIE formulations. First, the operational approach does not depend on the availability of the BIE for a given problem or the explicit expression of the fundamental solution. Second, the physical meaning (equilibrium of forces, moments, and so on) of these integral identities (Liu and Rudolphi 1991) can be identified easily from their derivations using this operational approach. Third, the identities derived from the operational approach are applicable to both the finite and infinite domain problems, while the simple solution approach can only be applied to a finite domain since the rigid-body motion cannot be imposed to an infinite domain directly.

The integral identities in (1)–(3) and (5)–(7) for the fundamental solutions have been employed successful in the development of the weakly-singular forms of both conventional and hypersingular BIEs for potential, elastostatic, acoustic, elastodynamic and electromagnetic problems (Rudolphi 1991; Liu and Rudolphi 1991; Liu and Rizzo 1992; Liu and Rizzo 1993; Muci-Kuchler and Rudolphi 1993; Chao, Liu et al. 1995; Liu and Chen 1999), for elastoplasticity (Poon, Mukherjee et al. 1998) and thermoelasticity (Mukherjee, Shah et al. 1999), and referenced by others (see, e.g., (Tanaka, Sladek et al. 1994; Johnston 1997)). Using these identities offers a general and systematic approach to the development of the weakly-singular forms of the BIEs, as compared to the earlier approach where the explicit expressions of the fundamental solutions need to be exploited in great length in order to cancel the singularities in the BIEs (see, e.g., (Rudolphi, Krishnasamy et al. 1988)).

The process developed in (Liu and Rudolphi 1991) to establish the identities for the fundamental solutions can be applied to obtain more identities which are only found useful recently. We will derive the following two new integral identities for the fundamental solutions of the potential and elastostatic problems and later show how to use them to establish the non-singular forms of the corresponding conventional BIEs in the next section:

The fourth identity for the potential problem

$$\int_S \frac{\partial G(P,P_o)}{\partial n}(x_k - x_{ok})\mathrm{d}S(P) - \int_S G(P,P_o)n_k(P)\mathrm{d}S(P) = 0, \quad \forall P_o \in V \cup E \ , \tag{9}$$

and the fourth identity for the elastostatic problem

$$\int_S T_{ij}(P,P_o)(x_k - x_{ok})\mathrm{d}S(P) - E_{jkpq}\int_S U_{ip}(P,P_o)n_q(P)\mathrm{d}S(P) = 0, \quad \forall P_o \in V \cup E \ . \tag{10}$$

As for the first three identities in (1)–(3) and (5)–(7), the domain V for these two new identities is an arbitrary (closed) domain in R^m. Therefore, S is an arbitrary (closed) surface in 3-D or curve in 2-D, which does not necessarily coincide with the real boundary of the problem domain under consideration (Liu and Rudolphi 1991).

To derive identity (9), we multiply Eq. (4) by $(x_k - x_{ok})$ and then integrate both sides over the domain V:

$$\int_V (x_k - x_{ok})G_{,jj}\,\mathrm{d}V(P) + \int_V (x_k - x_{ok})\delta(P,P_o)\mathrm{d}V(P) = 0, \quad \forall P_o \in V \cup E \ , \tag{11}$$

where $G_{,jj} = \partial^2 G/\partial x_j \partial x_j = \nabla^2 G$. By the sifting property of the Dirac-delta function (see, e.g., (Zemanian 1987)), the second integral in (11) vanishes and the first one can be evaluated as follows:

$$\begin{aligned}\int_V (x_k - x_{ok})G_{,jj}\,\mathrm{d}V(P) &= \int_V \left[(x_k - x_{ok})G_{,j}\right]_{,j}\mathrm{d}V(P) - \int_V (x_k - x_{ok})_{,j}G_{,j}\,\mathrm{d}V(P) \\ &= \int_S (x_k - x_{ok})G_{,j}n_j\,\mathrm{d}S(P) - \int_S \delta_{kj}Gn_j\,\mathrm{d}S(P) \\ &= \int_S (x_k - x_{ok})\frac{\partial G}{\partial n}\mathrm{d}S(P) - \int_S Gn_k\,\mathrm{d}S(P) \ ,\end{aligned}$$

where the Gauss' theorem in the sense of generalized functions has been applied (Liu and Rudolphi 1991). Substituting this result into (11), we obtain the fourth identity (9) for the potential problem.

To derive identity (10) for the elastostatic problem, we multiply Eq. (8) by $(x_l - x_{ol})$ and then integrate both sides over V:

$$\int_V (x_l - x_{ol})\Sigma_{ijk,k}(P,P_o)\mathrm{d}V(P) + \int_V (x_l - x_{ol})\delta_{ij}\delta(P,P_o)\mathrm{d}V(P) = 0, \quad \forall P_o \in V \cup E \ . \tag{12}$$

Again, the second integral vanishes due to the sifting property of the delta function. The first integral is evaluated below:

$$\begin{aligned}\int_V (x_l - x_{ol})\Sigma_{ijk,k}\,\mathrm{d}V(P) &= \int_V \left[(x_l - x_{ol})\Sigma_{ijk}\right]_{,k}\mathrm{d}V(P) - \int_V (x_l - x_{ol})_{,k}\Sigma_{ijk}\,\mathrm{d}V(P) \\ &= \int_S (x_l - x_{ol})\Sigma_{ijk}n_k\,\mathrm{d}S(P) - \int_V \delta_{kl}\left[E_{jkpq}U_{ip,q}\right]\mathrm{d}V(P) \\ &= \int_S (x_l - x_{ol})T_{ij}\,\mathrm{d}S(P) - \int_S E_{jlpq}U_{ip}n_q\,\mathrm{d}S(P) \ ,\end{aligned}$$

where $T_{ij} = \Sigma_{ijk}n_k$ and $\Sigma_{ijk} = E_{jkpq}U_{ip,q}$ have been applied. Substituting the above result into (12) and switching the subscript l to k, we obtain the fourth identity (10) for the elastostatic problem.

Note that in establishing these integral identities, the only relations one has to exploit are the original governing differential equations for the fundamental solutions and the properties of the Dirac-delta functions representing the unit, concentrated source. One does not need to know the explicit expressions of the fundamental solutions, or the related BIE formulations of the problems. In fact, more integral identities for the fundamental solutions can be derived readily and systematically using the process described above and in (Liu and Rudolphi 1991), if they are deemed useful in the applications of the boundary element method. In general, all the integral identities for the fundamental solution of the potential problem can be derived by starting with the following integration of the governing equation:

$$\int_V (x_k - x_{ok})^\alpha \frac{\partial^\beta}{\partial x_{oi}^\beta}\left[\nabla^2 G + \delta(P,P_o)\right]\mathrm{d}V(P) = 0, \quad \forall P_o \in V \cup E \ ,$$

where $\alpha,\ \beta = 0, 1, 2, 3, \ldots$. A similar starting integral expression exists for the elastostatic problem. The first four identities we have derived so far are corresponding to the combinations of $\alpha,\ \beta = 0$ or 1.

The two new identities (9) and (10) for the fundamental solutions of the potential and elastostatic problems, respectively, can also be derived by using the simple solution idea as developed in Ref. (Rudolphi 1991). This will be discussed in the next Section after the corresponding BIEs are presented. These two identities will be employed to establish the non-singular forms of the conventional BIEs in the next section, just as the case that the identities (1)–(3) and (5)–(7) have been applied readily to develop the weakly-singular forms of the conventional and hypersingular BIEs (Liu and Rudolphi 1991).

3 The non-singular forms of the conventional BIEs

3.1 Potential problem

We start with the following integral representation for the potential problem:

$$\phi(P_o) = \int_S \left[G(P,P_o)\frac{\partial\phi(P)}{\partial n} - \frac{\partial G(P,P_o)}{\partial n}\phi(P)\right]\mathrm{d}S(P), \quad \forall P_o \in V \ , \tag{13}$$

where ϕ is the unknown potential satisfying the Laplace equation $\nabla^2\phi = 0$, $G(P,P_o) = 1/(4\pi r)$ or $(1/2\pi)\ln(1/r)$ (with r being the distance from P_o to P, Fig. 1) is the fundamental solution for 3-D or 2-D problem, respectively. It is interesting to note that if we impose the following simple (linear) solution

$$\phi(P) = d_k(x_k - x_{ok}) \ ,$$

where d_k are arbitrary constants, in the above integral representation, the new identity (9) is recovered, as expected.

Direct limit process as $P_o \to S$ in Eq. (13) leads to the conventional BIE for the potential problem which contains both weakly- and strongly-singular integrals. This singular form of the BIE can be converted into the following weakly-singular form readily using the integral identity (1) (see, e.g., (Liu and Rudolphi 1991)):

$$\int_S \frac{\partial G(P,P_o)}{\partial n}[\phi(P) - \phi(P_o)]\mathrm{d}S(P) = \int_S G(P,P_o)\frac{\partial\phi(P)}{\partial n}\mathrm{d}S(P), \quad \forall P_o \in S \ ,$$

for a finite domain, in which both integrals are weakly-singular.

To further regularize the singular integral, we use the following two-term subtraction for the second integral in (13):

$$\begin{aligned}\int_S \frac{\partial G(P,P_o)}{\partial n}\phi(P)\mathrm{d}S(P) &= \int_S \frac{\partial G(P,P_o)}{\partial n}[\phi(P) - \phi(P_o) \\ &\quad - \phi_{,k}(P_o)(x_k - x_{ok})]\mathrm{d}S(P) + \phi(P_o)\int_S \frac{\partial G(P,P_o)}{\partial n}\mathrm{d}S(P) \\ &\quad + \phi_{,k}(P_o)\int_S \frac{\partial G(P,P_o)}{\partial n}(x_k - x_{ok})\mathrm{d}S(P) \\ &= \int_S \frac{\partial G(P,P_o)}{\partial n}[\phi(P) - \phi(P_o) \\ &\quad - \phi_{,k}(P_o)(x_k - x_{ok})]\mathrm{d}S(P) - \phi(P_o) \\ &\quad + \phi_{,k}(P_o)\int_S G(P,P_o)n_k(P)\mathrm{d}S(P), \quad \forall P_o \in V \ ,\end{aligned}$$

where the first identity (1) and the new fourth identity (9) have been applied. Substituting this result into the integral representation (13) and rearranging the terms, we obtain

$$\int_S \frac{\partial G(P,P_o)}{\partial n}\left[\phi(P)-\phi(P_o)-\phi_{,k}(P_o)(x_k-x_{ok})\right]\mathrm{d}S(P)$$
$$=\int_S G(P,P_o)\left[\phi_{,k}(P)-\phi_{,k}(P_o)\right]n_k(P)\mathrm{d}S(P),$$
$$\forall P_o\in S\ , \tag{14}$$

where the limit of $P_o \to S$ has been taken. This is the non-singular form of the conventional BIE for the potential problem in a finite domain. For an infinite domain, the free term $\phi(P_o)$ will appear on the left-hand side of Eq. (14) (cf., (Liu and Rudolphi 1991)). The two integrals in Eq. (14) are regular even when $P \to P_o (r \to 0)$ because the singularities can be cancelled completely according to the following estimates of the orders for all the terms (in 3-D):

$$\frac{\partial G(P,P_o)}{\partial n}\sim \mathrm{O}\left(\frac{1}{r^2}\right),$$
$$\left[\phi(P)-\phi(P_o)-\phi_{,k}(P_o)(x_k-x_{ok})\right]\sim \mathrm{O}(r^2),$$
$$G(P,P_o)\sim \mathrm{O}\left(\frac{1}{r}\right),$$
$$\left[\phi_{,k}(P)-\phi_{,k}(P_o)\right]\sim \mathrm{O}(r),\quad \text{as } r\to 0\ ;$$

assuming that $\phi(P)$ has continuous first derivatives in the neighborhood of the source (collocation) point P_o. A discretization procedure similar to that as described in (Liu and Rizzo 1992) can be employed in the discretization of Eq. (14).

3.2
Elastostatic problem

The integral representation of the displacement field for the elastostatic problem is

$$u_i(P_o)=\int_S\left[U_{ij}(P,P_o)t_j(P)-T_{ij}(P,P_o)u_j(P)\right]\mathrm{d}S(P),$$
$$\forall P_o\in V\ , \tag{15}$$

where u_i and t_i are the displacement and traction fields, respectively. Again, if we impose the following simple solution (rotation fields)

$$u_i(P)=d_{ik}(x_k-x_{ok})\ ,$$

where d_{ik} are arbitrary constants, to the integral representation (15), the new identity (10) for elastostatics is recovered.

Similar to the potential problem, the following weakly-singular form of the conventional BIE

$$\int_S T_{ij}(P,P_o)\left[u_j(P)-u_i(P_o)\right]\mathrm{d}S(P)$$
$$=\int_S U_{ij}(P,P_o)t_j(P)\mathrm{d}S(P),\quad \forall P_o\in S\ ,$$

for a finite domain can be derived from (15) easily using the first identity (5) (see (Cruse 1974; Rizzo and Shippy 1977) and (Liu and Rudolphi 1991)).

To derive the non-singular form of the BIE, we use the two-term subtraction for the density function in the strongly-singular integral in (15) as follows:

$$\int_S T_{ij}(P,P_o)u_j(P)\mathrm{d}S(P)$$
$$=\int_S T_{ij}(P,P_o)\left[u_j(P)-u_j(P_o)-u_{j,k}(P_o)(x_k-x_{ok})\right]\mathrm{d}S(P)$$
$$+u_j(P_o)\int_S T_{ij}(P,P_o)\mathrm{d}S(P)$$
$$+u_{j,k}(P_o)\int_S T_{ij}(P,P_o)(x_k-x_{ok})\mathrm{d}S(P),\ \ \forall P_o\in V\ .$$

Applying the first identity (5) and the fourth identity (10) in the above expression, we have

$$\int_S T_{ij}(P,P_o)u_j(P)\mathrm{d}S(P)$$
$$=\int_S T_{ij}(P,P_o)[u_j(P)-u_j(P_o)-u_{j,k}(P_o)(x_k-x_{ok})]\mathrm{d}S(P)$$
$$-u_i(P_o)+u_{j,k}(P_o)E_{jkpq}\int_S U_{ip}(P,P_o)n_q(P)\mathrm{d}S(P)$$
$$=\int_S T_{ij}(P,P_o)[u_j(P)-u_j(P_o)-u_{j,k}(P_o)(x_k-x_{ok})]\mathrm{d}S(P)$$
$$-u_i(P_o)+\sigma_{jk}(P_o)\int_S U_{ij}(P,P_o)n_k(P)\mathrm{d}S(P),\ \ \forall P_o\in V\ ,$$

where $E_{jkpq}u_{j,k}=E_{pqjk}u_{j,k}=\sigma_{pq}$ has been used in the last step. Substituting this result into (15), rearranging the terms and letting the source point $P_o \to S$, we obtain the following non-singular form of the conventional BIE for the elastostatic problem:

$$\int_S T_{ij}(P,P_o)\left[u_j(P)-u_j(P_o)-u_{j,k}(P_o)(x_k-x_{ok})\right]\mathrm{d}S(P)$$
$$=\int_S U_{ij}(P,P_o)\left[\sigma_{jk}(P)-\sigma_{jk}(P_o)\right]n_k(P)\mathrm{d}S(P),$$
$$\forall P_o\in S\ , \tag{16}$$

which is valid for a finite domain. A free term $u_i(P_o)$ needs to be added to the left-hand side of Eq. (16) if it is applied to an infinite domain. Similar to the case of the potential problem, singularities in the two kernels U_{ij} and T_{ij} are cancelled out due to the use of Taylor's series expansions for the density functions.

A striking phenomenon about the weakly-singular and non-singular BIE formulations, as derived in (Liu and Rudolphi 1991) and above using the identities for the fundamental solutions, is that the two integrals are regularized to weakly- or non-singular integrals at the same time, as is also shown in Ref. (Cruse and Richardson 1996). The properties of the fundamental solutions, as represented by the identities, play an important role in achieving this regularization. Weakly-singular, strongly-singular, or hypersingular integrals will be cancelled out naturally, and completely, from both sides of the BIE formulations by exploiting these identities. All these results reveal the non-singular or weakly-singular nature of the BIE formulations for the physical problems which, in most cases, are not singular at all in the first place. The singularity in the fundamental solutions, which has hindered the BEM research ever since its beginning, is deceiving and should not lead to the singularities of the BIE formulations, if the properties of the fundamental solutions have been examined and utilized.

4
Implication to the smoothness requirement for the BIEs

It is now well known that the theory imposes certain continuity requirement on the density functions in the boundary integral equations, in order for the limits of the integrals as the source point approaching the boundary to exist (Krishnasamy, Rizzo et al. 1992; Martin and Rizzo 1996). For example, in the case of potential problems, the density function $\phi(P)$ must be $C^{0,\alpha}$ continuous for the limit of the strongly-singular integral

$$\lim_{P_o \to S} \int_S \frac{\partial G(P, P_o)}{\partial n} \phi(P) \mathrm{d}S(P)$$

to exist, and must be $C^{1,\alpha}$ continuous for the limit of the hypersingular integral

$$\lim_{P_o \to S} \int_S \frac{\partial^2 G(P, P_o)}{\partial n \partial n_o} \phi(P) \mathrm{d}S(P)$$

(n_o is the normal at the source point) to exist. This smoothness requirement means that the density function $\phi(P)$, or its derivatives, must be Hölder continuous in the neighborhood of the source point P_o in order for the strongly-singular, or hypersingular, integrals to be meaningful, respectively. This requirement imposes severe limitations to the applications of BIEs. For example, this smoothness requirement will exclude, theoretically, the use of C^0 boundary elements, such as the conforming quadratic elements, in the discretizations of hypersingular BIEs which have been found very useful for many problems in applied mechanics. Relaxation of this smoothness requirement for the hypersingular BIEs, in either strongly-singular or weakly-singular forms and with conforming C^0 elements, have been attempted by several authors (see, e.g., (Chien, Rajiyah et al. 1990; Wu, Seybert et al. 1991; Liu and Rizzo 1992; Cruse and Suwito 1993; Huang and Cruse 1994; Cruse and Richardson 1996)). The validation of this relaxation has also been provided (Richardson, Cruse et al. 1997; Liu and Chen 1999; Martin, Rizzo et al. 1998) due to the strong need to do this in the BEM community. It has been postulated in Ref. (Liu and Chen 1999) that the original $C^{1,\alpha}$ continuity requirement on the density function in the hypersingular BIE formulations can be relaxed to piecewise $C^{1,\alpha}$ continuity in the numerical implementation of the weakly-singular forms of the hypersingular BIEs. This relaxation means that conforming linear, quadratic, and other higher-order elements, as well as nonconforming elements (including the constant elements), can be applied to the weakly-singular forms of the hypersingular BIEs.

The existence of the non-singular forms of the conventional BIEs raises perplexing, and yet intriguing, questions about the smoothness requirement and its relaxations. From the derivations of the non-singular forms of the conventional BIEs for both potential and elastostatic problems, as described in the previous section, it is obvious that the density functions must have continuous first derivatives in order for the non-singular forms to be meaningful. This means that, in theory, the smoothness requirement for the density functions is, in fact, tightened from $C^{0,\alpha}$ to $C^{1,\alpha}$ continuity for the non-singular forms of the conventional BIEs. If we adhere to this tightened smoothness requirement in the discretizations of the non-singular forms of the conventional BIEs, we would have to use nonconforming elements or C^1elements. This is certainly undesirable as in the case of the hypersingular BIEs. However, if we adopt the relaxation strategy used for hypersingular BIEs, that is, relaxing the smoothness requirement to piecewise $C^{1,\alpha}$ continuity in the discretizations (Liu and Chen 1999), the dilemma between the theory and the application can be avoided. Similar to the argument in Ref. (Liu and Chen 1999), computation of the integrals on an element and containing the two-term subtraction of the density functions, as in Eqs. (14) and (16), only requires that the density functions are $C^{1,\alpha}$ continuous on that element. This leads to the argument of the piecewise continuity in discretizations of the various BIEs. For example, with this relaxation, we can still apply C^0 elements, such as the conforming linear and quadratic elements, in the discretizations of the non-singular forms of the conventional BIEs. Again, the convergence study, either analytical or numerical, will be crucial in the validation of this piecewise continuity argument (Liu and Chen 1999). Numerical studies for 2-D elastostatic and 3-D acoustic problems have shown that this relaxation can lead to converged results for the weakly-singular forms of the hypersingular BIEs with conforming C^0 elements (see, (Richardson, Cruse et al. 1997; Liu and Chen 1999)).

From the above discussions, the following implications from the existence of the non-singular BIE formulations can be drawn: (1) Singular integrals, including weakly-singular, strongly-singular and hypersingular ones, in the BIE formulations can be removed completely by using the identities for the fundamental solutions which are the original source of the singularity. (2) The smoothness requirement (C^1 continuity) for the density functions in the non-singular forms of the conventional BIEs is higher than (thus tightened from) that (C^0) for the original singular or weakly-singular forms of the BIEs. However, if the relaxation strategy (Richardson, Cruse et al. 1997; Liu and Chen 1999; Martin, Rizzo et al. 1998) is adopted in the discretizations of the BIEs in these forms, the tightened smoothness requirement may not necessarily hinder the applications of such BIE formulations. (3) Convergence study of the relaxation strategies is crucial and urgently needed in order to finally settle the smoothness issue in the BIE/BEM applications.

5
Conclusion

Two new integral identities for the fundamental solutions of the potential and elastostatic problems have been established in this paper, based on a general, operational approach which does not depend on the corresponding BIEs. Non-singular forms of the conventional BIEs for both potential and elastostatic problems have been developed by using these identities. The existence of these non-singular forms of the conventional BIEs raises an interesting question to the smoothness requirement in the BIE formulations, that is, smoothness requirement for the non-singular BIEs is in fact increased from C^0 to C^1 continuity for the density functions, due to the two-term subtractions. A feasible remedy to this dilemma may be to adopt the relaxation strategy, that is, to reduce the smoothness requirement to that in the piecewise sense in the discretizations of the non-singular BIEs. However, a convergence study, especially a theoretical one, or a counter-example showing divergence, for the relaxation strategy is still urgently needed before the smoothness issue in the BEM can be finally settled.

Numerical studies of the developed non-singular conventional BIEs will be conducted to investigate their possible advantages over other forms of the BIEs. Extension of the established approach to develop non-singular forms of the conventional BIEs for other problems and the non-singular forms of the hypersingular BIEs is readily achievable, if these forms are found useful in the BEM applications.

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