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# Slow convergence of the BEM with constant elements in solving beam bending problems

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ABSTRACT

Constant elements offer many advantages as compared with other higher-order elements in the boundary element method (BEM). With the use of constant elements, integrals in the BEM can be calculated accurately with analytical integrations and no corner problems need to be addressed. These features can make fast solution methods for the BEM (such as the fast multipole, adaptive cross approximation, and pre-corrected fast Fourier transform methods) especially efficient in computation. However, it is well known that the collocation BEM with constant elements is not adequate for solving beam bending problems due to the slow convergence or lack of convergence in the BEM solutions. In this study, we quantify this assertion using simple beam models and applying the fast multipole BEM code so that a large number of elements can be used. It is found that the BEM solutions do converge numerically to analytical solutions. However, the convergence rate is very slow, in the order of $h$ to the power of 0.55–0.63, where $h$ is the element size. Some possible reasons for the slow convergence are discussed in this paper.


## 1. Introduction

The constant elements have been applied in the boundary element method (BEM) to solve elasticity problems ever since the inception of the method in the early 1960s [1,2]. Constant elements are introduced in almost of all the textbooks on the BEM [3–7]. However, it has long been speculated that the BEM with constant elements converge very slowly for solving beam bending problems or it may not converge at all for such problems. This can cause problems in testing an elasticity BEM code. With the same code, satisfactory BEM results can be obtained when bulky-shaped domains are considered, while poor results are observed with no obvious reasons when slender structures applied with bending loads are considered. With the conventional BEM, the model size is limited to a few thousands of elements and thus the convergence study cannot be conducted fully on a desktop computer. This may be the main reason that no detailed results have been reported in the literature on the convergence of the collocation BEM with constant elements for solving beam bending problems.

With the developments of the fast multipole, adaptive cross approximation, and other fast solution methods for the BEM [8], we are now able to conduct detailed numerical convergence study on the constant element BEM for solving beam bending problems with the number of elements close to one million on a desktop computer. Therefore, it is possible now to answer the convergence question with some numerical studies about the elasticity BEM with constant elements in solving beam bending problems.

This paper presents some BEM results with large-scale 2D models of simple beam bending problems to provide some quantitative answers to the slow convergence question regarding the constant element BEM for beam bending problems. Discussions on possible reasons of this slow convergence are also provided based on the numerical results. Some conclusions are drawn based on the numerical results.

## 2. Setup of the study

The BEM is based on the following direct boundary integral equation (BIE) for general 2D elastostatic problems [3–7]:

$$\frac{1}{2}u_i(\mathbf{x}) = \int_S [U_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{y}) - T_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{y})]dS(\mathbf{y}), \quad \forall \mathbf{x} \in S, \tag{1}$$

where $u_i$ and $t_i$ are the displacement and traction, respectively; $S$ is the boundary of domain $V$; $S$ is assumed to be smooth around $\mathbf{x}$; and $i$, $j = 1$, 2 in 2D cases. The two kernel functions $U_{ij}(\mathbf{x},\mathbf{y})$ and $T_{ij}(\mathbf{x},\mathbf{y})$ are the displacement and traction components, respectively, in the fundamental solution (also called Kelvin's solution) as given in Refs. [3–7]. The $U$ kernel is weakly singular, while the $T$ kernel is strongly singular requiring the integral to be evaluated in the sense of Cauchy-principal value (CPV). With constant elements, all the

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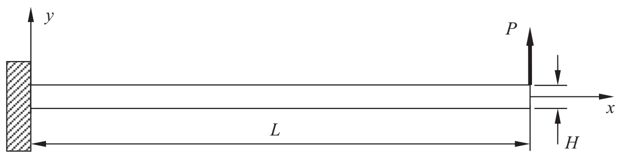


**Fig. 1.** A cantilever beam with a point load at the end.

integrals (including the CPV integrals) with these two kernels can be computed analytically in the BEM [7].

In this study, the fast multipole BEM code for 2D elasticity problems reported in Refs. [7,9] is used to solve BIE (1). In this 2D elasticity code, constant elements are used and all the integrals of the kernels and moments in the fast multipole BEM are evaluated analytically [7]. Therefore, there are no difficulties with the nearly singular integrals when the domain is thin as in the case of a beam. This fast multipole BEM code has been tested extensively for 2D stress analysis and consistently delivers the same results as with the conventional BEM code. Very good results have been obtained for stress concentration problems on regular domains (e.g., a square plate with a hole at the center) [7,9].

The beam bending problem studied is sketched in Fig. 1. The cantilever beam has a length $L$, height $H$, width $B$ (not shown), and a lateral point force $P$ applied at the free end. In this study, we assume that $H=1$, $B=1$, $P=1$, Young's modulus $E=1$, and Poisson's ratio $\nu=0.3$. Three cases are studied with the length ratio $L/H=5$, 10 and 20. Based on the 2D elasticity theory (plane stress), the analytical solution of the maximum deflection $w_{\max}$ at ($L$, 0) location is given in Ref. [10] (see Section 21):

$$w_{\max}=\frac{PL^3}{3EI}+\frac{PH^2L}{8GI} \tag{2}$$

where $I$ is the second moment of area (or the moment of inertia of the cross sectional area), and $G$ is the shear modulus. This analytical solution accounts for the shear deformation and is therefore valid for both short and slender beams as the ones studied here.

We start the BEM mesh with the element size=1 and then reduce the element size repeatedly until the element size reaches 0.0001, where the BEM results for all the three cases ($L/H=5$, 10, 20) give the relative errors around or below 1% for the maximum deflection. For the case $L/H=20$, the smallest BEM model has 42 elements and 84 DOFs (degrees of freedom), while the largest BEM model has 420,000 elements and 840,000 DOFs.

## 3. Numerical results

Fig. 2 shows the relative error in the computed maximum deflection (at $x=L$, $y=0$) for the three cases ($L/H=5$, 10, 20) using the BEM. As the number of elements increases, the errors in the BEM results do reduce, albeit at a very slow fashion. This is especially true for the slender beam case with $L/H=20$, for which the slowest convergence is observed and the bending behavior dominates. For the error to be near or below 10%, it took a model with 1200 DOFs for the case with $L/H=5$, with 8800 DOFs for the case of $L/H=10$, and with 42,000 DOFs for the case of $L/H=20$. When the number of DOFs reaches 840,000, the error reduces to 0.3% for the case of $L/H=20$, which is considered extremely slow for convergence with the BEM.

To see the convergence rate of the BEM with constant elements in the three cases, we plot the relative errors in the computed maximum deflection against the element size, as shown in Fig. 3. The plots are in log–log scale, so that the slopes of the trendlines of the curves represent the convergence rates, which are also shown in the inserted formulas. It can be seen that for all the three cases,

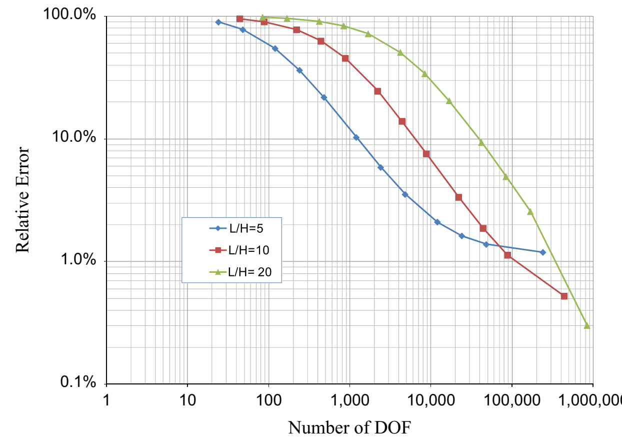


**Fig. 2.** Relative errors of the maximum deflections for the three beam models.

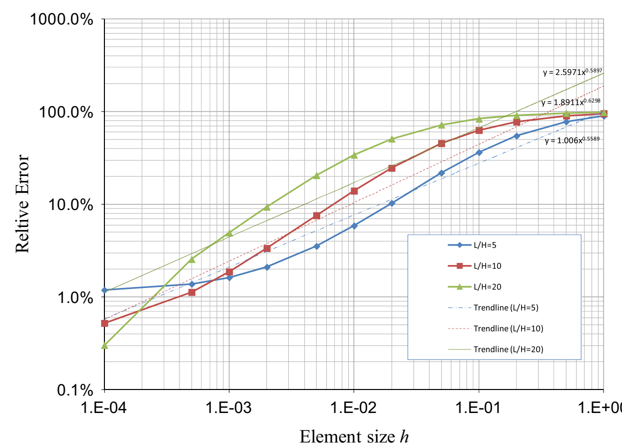


**Fig. 3.** Convergence rates of the BEM solutions for the three beam models.

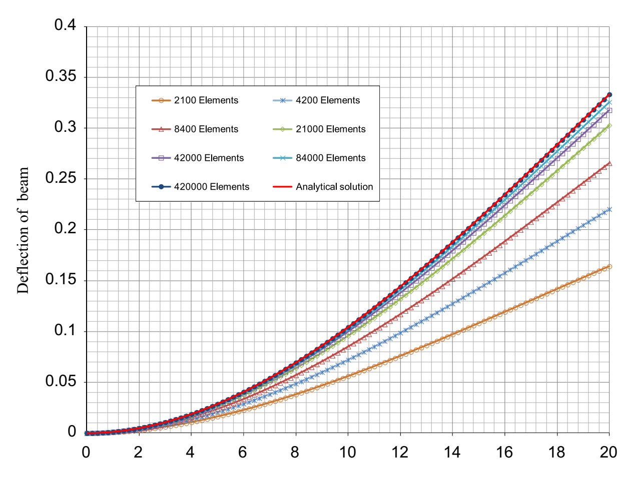


**Fig. 4.** Plot of the deflection computed using the BEM for the model with $L/H=20$.

the convergence rates are between $O(h^{0.55})$ and $O(h^{0.63})$, where $h$ is the element size.

In Fig. 4, we plot the deflection along the axis of the beam and compare the BEM results with the analytical one (the values are dimensionless). Again, large discrepancies in the BEM results and the analytical solution are observed. Acceptable results are achieved only when the number of elements reaches 420,000 (the number of DOFs=840,000), where the BEM curve finally approaches to the analytical solution.

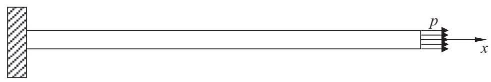


**Fig. 5.** The same beam with $L/H=20$ and under a tension load (the case of a bar).

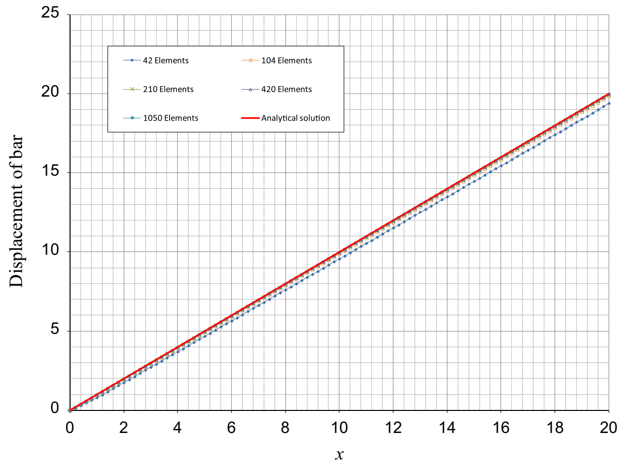


**Fig. 6.** BEM results of the $x$-displacement of the bar under tension.

To confirm that this slow convergence of the BEM with constant elements occurs only with beam bending type of problems, we use the same beam model with the length ratio $L/H=20$ and change the load to a tension load, as shown in Fig. 5, where the negative pressure load $p$ is set as 1. All the other parameters used in the bending case remain the same. Then the axial displacement along the beam (should be called a bar now) is calculated and compared with the analytical solution. The results are shown in Fig. 6. With only 42 elements, the relative error in the maximum $x$-displacement (total elongation of the bar) reaches 3.0%. The BEM results improve quickly, with the error reaching 0.10% for the largest BEM model with 1050 constant elements. The convergence rate is found to be $O(h^{1.05})$ in this case. This additional test confirms that the BEM with constant elements works fine for non-bending type of problems.

## 4. Discussions

To find out the cause of the slow convergence of the BEM solutions, we study the conditioning of the linear systems of equations for the beam-bending and bar-in-tension cases. Using the models with the aspect ratio $L/H=20$, the condition numbers of the BEM systems for the beam and bar cases are computed and plotted in Fig. 7 when the number of elements increases from 42 to 12,600. The condition numbers are identical for the beam and bar cases because the coefficient matrices are the same in the two cases. As shown in Fig. 7, the condition numbers are in the order of $10^4$–$10^5$. These values are relatively high, nevertheless are stable. Using double precision in the code, good BEM results have been obtained with the systems of equations having the condition numbers in the order of $10^8$ [11]. In this study, all the fast multipole BEM solutions were obtained within 100 iterations using the GMRES solver for a tolerance of $10^{-4}$, for beam models with all the three aspect ratios. This indicates that the conditioning of BEM systems of the equations does not cause the accuracy problem. This is also indicated by the good results, as shown in Fig. 6, when the bar case is considered (the same system of equations is solved with a different right-hand side vector).

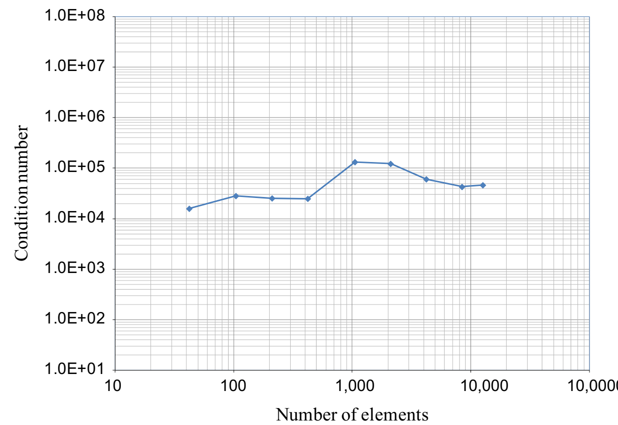


**Fig. 7.** Condition numbers of the systems of equations for the beam and bar models.

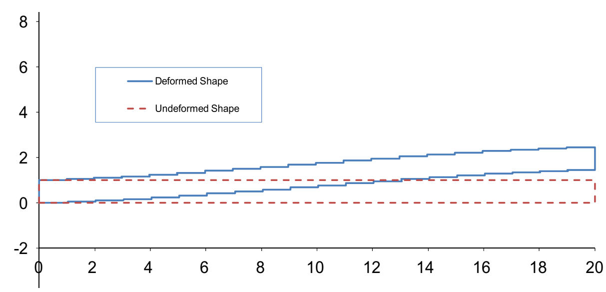


**Fig. 8.** Deformed shape of the beam model under bending load (42 elements, scaled).

Therefore, conditioning of the BEM equations can be excluded from the causes of the slow convergence in the beam case.

One may ask if the BEM system of equations becomes degenerated when the domain is thin, which may lead to the slow convergence of the BEM results. This is not the case either since the largest aspect ratio $L/H=20$ is well below the definition of a thin shape or thin domain (for which, the aspect ratio should be above 100). Even when the aspect ratio is above 100, it is shown in Ref. [12] that the BIE does not degenerate for interior problems, contrary to the case of a crack-like problem [13]. When the thickness of the domain is finite, the BIE gives two distinct equations at the two opposing collocation points, one on the top surface and one on the bottom surface of the thin domain. Therefore, the BEM is in general adequate for solving thin shape or beam/shell-like structures, if the nearly singular integrals are computed accurately [12,14].

The slow convergence of the BEM for beam bending problems is most likely due to the characteristics of the constant elements, not the conditioning or degeneracy of the BEM systems. Specifically, constant elements cannot describe adequately the bending shape of a beam, due to the fact that constant elements cannot represent the rigid-body rotation (which requires a linear term in the displacement expression). This can cause errors in the bending case when the rigid-body rotation and shear deformation dominate on each element. In fact, when the beam is in bending, the solution obtained by using constant elements gives zigzagged (stepped) curves for the deformed top and bottom boundaries of the beam (see Fig. 8). This leads to large errors in the calculation of the displacement and strain fields, unless the size of elements is extremely small as we have shown in this study. On the other side, this situation is not so severe when the same beam is applied with a tension load, where rigid-body translation dominates on each

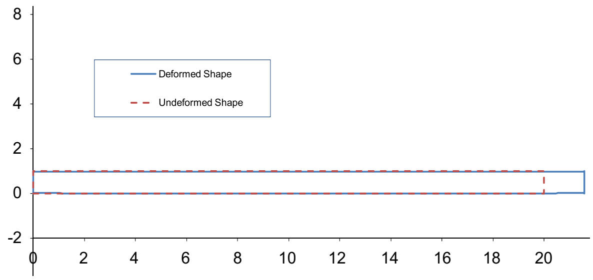


**Fig. 9.** Deformed shape of the bar model under tension (42 elements, scaled).

element and the constant element can represent this translation correctly (see Fig. 9).

It should be pointed out that the slow convergence with constant elements shown in this study does not exist when linear or quadratic elements are used for the beam bending problems. For example, there is a beam bending example solved in Ref. [3] (Chapter 4, Example 4.4) which is almost identical to our case with $L/H=5$ (The only difference is that a distributed shear load is applied at the end of the free end, while in our case a concentrated load is applied.). With only 12 quadratic elements, the relative error in the maximum deflection is less than 1% compared with the analytical solution. In fact, it is also mentioned in Ref. [3] that constant elements are not adequate for solving such beam bending problems, without giving further explanations.

## 5. Conclusion

Slow convergence of the collocation BEM with constant elements in solving beam bending problems is reported in this paper based on the BEM results for three cantilever beam models with different aspect ratios. Although the BEM results converge to the analytical solutions, the convergence rates are found to be very slow and are in the range of $O(h^{0.55})$–$O(h^{0.63})$. The main reason for this slow convergence is due to the behavior of the constant elements which cannot represent the rigid-body rotation of the displacement field correctly. Further theoretical studies are needed to prove the convergence of the collocation BEM with constant elements.

## Acknowledgments

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## References

[1] Jaswon MA. Integral equation methods in potential theory. I. Proc R Soc Lond A 1963;275:23–32.
[2] Rizzo FJ. An integral equation approach to boundary value problems of classical elastostatics. Q Appl Math 1967;25:83–95.
[3] Brebbia CA, Dominguez J. Boundary elements – an introductory course. New York: McGraw-Hill; 1989.
[4] Banerjee PK. The boundary element methods in engineering. 2nd ed. New York: McGraw-Hill; 1994.
[5] Kane JH. Boundary element analysis in engineering continuum mechanics. Englewood Cliffs, NJ: Prentice Hall; 1994.
[6] Becker AA. The boundary element method in engineering. New York: McGraw-Hill; 1992.
[7] Liu YJ. Fast multipole boundary element method – theory and applications in engineering. Cambridge: Cambridge University Press; 2009.
[8] Liu YJ, Mukherjee S, Nishimura N, Schanz M, Ye W, Sutradhar A, et al. Recent advances and emerging applications of the boundary element method. Appl Mech Rev 2011;64:1–38.
[9] Liu YJ. A new fast multipole boundary element method for solving large-scale two-dimensional elastostatic problems. Int J Num Methods Eng 2006;65:863–81.
[10] Timoshenko SP, Goodier JN. Theory of elasticity. 3rd ed. New York: McGraw-Hill; 1987.
[11] Liu YJ. Dual BIE approaches for modeling electrostatic MEMS problems with thin beams and accelerated by the fast multipole method. Eng Anal Bound Elem 2006;30:940–8.
[12] Liu YJ. Analysis of shell-like structures by the boundary element method based on 3-D elasticity: formulation and verification. Int J Num Methods Eng 1998;41:541–58.
[13] Krishnasamy G, Rizzo FJ, Liu YJ. Boundary integral equations for thin bodies. Int J Num Methods Eng 1994;37:107–21.
[14] Luo JF, Liu YJ, Berger EJ. Analysis of two-dimensional thin structures (from micro- to nano-scales) using the boundary element method. Comput Mech 1998;22:404–12.