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A fast multipole boundary element method for modeling 2-D multiple crack problems with constant elements

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ABSTRACT

A fast multipole boundary element method (BEM) for solving 2-D multiple crack problems in linear elastic fracture mechanics is presented in this paper. For multiple crack problems, both the degrees of freedom (DOFs) and the size of system matrices increase quickly as the number of cracks increases, and the conventional BEM cannot support such large systems. Instead of using the singular quarter-point boundary elements at the crack tips, constant line elements are applied to symmetrically discretize the outer boundaries and crack surfaces in the present approach. In order to keep the accuracy within a limited acceptable range, a relatively large number of constant elements are required to discretize the crack surfaces. The crack opening displacement (COD) fields of the multiple crack problems are obtained by the fast multipole BEM. Stress intensity factors (SIFs) are extracted from the obtained displacement fields near the crack tip by using one point COD formula. Comparison of the CODs between the fast multipole BEM and a finite element method using ANSYS are illustrated to show the feasibility of the proposed approach. With the acceleration of fast multipole technique, multi-crack problems can be dealt with desktop PCs. Several numerical examples are presented for computing the SIFs of cracks to study the effectiveness and the efficiency of the proposed approach. The numerical results clearly demonstrate the potentials of the fast multipole BEM for solving 2-D large-scale multi-crack problems by using constant elements.

1. Introduction

Engineering materials and structures often contain multiple cracks. Cracks can easily lead to the failure of structures. Therefore, the prediction of crack behaviors and fracture characteristics of these solid materials with multiple cracks need to be studied. A major result of linear elastic fracture mechanics (LEFM) theory is that crack behaviors can be determined by the value of stress intensity factors (SIFs) at the crack tips, which is dependent on the applied load, crack size and geometrical configuration of the cracked structure. The computation of SIFs thus plays a very important role in LEFM applications. However, due to the huge consumption of computation time and computer memory, many multiple crack problems have not been solved numerically. More reliable, accurate and efficient numerical techniques are needed to develop a useful tool for modeling multiple crack problems.

The finite element method (FEM) has a well-documented history in fracture mechanics applications. There are fairly developed software, such as ANSYS and ABAQUS, now widely available in many engineering fields. In spite of the FEM wide-spread popularity, the multiple crack problems are probably one class of the most difficult problems to deal with using the FEM because of the discretization near the crack tips. The subsequent boundary element method (BEM) is also recognized as a highly efficient numerical technique for the LEFM analysis [1–3] because of its boundary-only discretization nature. In other words, only the outer boundaries of the domain and the crack surfaces need to be discretized. Compared with domain methods such as the FEM, the BEM reduces the initial data preparation because of the reduction in dimensionality of original problems. Furthermore, the remodeling process is much more convenient than the FEM. As a result, the conventional BEM enjoyed a good reputation of dimension reduction and easy meshing in remodeling. However, for the conventional BEM, when the model has millions of DOFs for large-scale multiple crack problems, it is inefficient for modeling the crack behaviors and their propagations on desktop PCs. Therefore, the computation efficiency and memory storage have been serious concerns for analyzing large-scale multiple crack problems using the BEM because of its dense and asymmetric coefficient matrices.

In order to reduce the CPU time and memory storage, the fast multipole method (FMM) pioneered by Greengard and Rokhlin [4]

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has been applied to the BEM in the mid of 1980s, leading to the so-called fast multipole BEM. The fast multiple BEM and its applications has been developed intensively by many researchers in the last two decades for elasticity problems. Peirce and Napier [5] developed a spectral multipole approach for 2-D large scale elastostatics analysis. Fu et al. [6] and Popov et al. [7] studied 3-D elastostatics problems. Yoshida et al. [8] applied a new fast multipole technique for solving 3-D elastostatic crack problems. Wang and Yao [9,10] applied a fast multipole BEM using a new form of complex Taylor series expansions and expressive results were obtained using higher-order elements for 2-D problems. Wang and Yao [9] also studied 2-D crack problems using a dual BIE approach with the CBIE collocating on one crack surface while HBIE on the opposite crack surface, which demonstrated the promises of the fast multipole BEM for solving large-scaled engineering problems. A comprehensive review of the fast BEM was elaborated by Nishimura in [11] and Liu in [12].

For a 2-D crack problem, in order to achieve better results of the displacement fields, higher-order elements and special crack tip elements, such as quarter-point elements have been often used in most previous publications, showing the effectiveness of using quarter-point elements. However, instead of using this special quarter-point elements at the crack tips, constant elements are applied to discretize the outer boundaries of the domains and crack surfaces in this work. Correspondingly, to keep the computing accuracy within an acceptable range, a relatively large number of constant elements are required on the crack surfaces. On the other hand, the size of the system matrices will grow very large with the increase of the number of cracks. Multiple crack problems usually need large BEM models while constant elements are used, possibly millions of degrees of freedom (DOFs) in some large-scale cases, which can lead to a time-consuming calculation. Based on the above two considerations, the fast multipole BEM is an efficient technique and can be employed to deal with large-scale cracks in case of the solution complexity increases.

In the fast multipole BEM for this study, a dual boundary integral equation (BIE) formation (a linear combination of conventional BIE (CBIE) and hypersingular BIE (HBIE)) is employed to analyze multiple crack problems by using constant elements based on the work [12]. First, in the fast multipole BEM models, similar to conventional BEM, the outer boundaries of the domain and crack surfaces are discretized by constant elements. Then, the multipole and local expansions of the kernel functions were used and the generalized minimal residual method (GMRES) is applied as the iterative solver to obtain the unknowns on the boundaries. With the application of fast multipole technique, both the CPU time and memory storage in most cases can be reduced to about $O(N)$, where N is the number of DOFs. Finally, once all the CODs are known, the values of stress intensity factors (SIFs) using the one point COD formula near the crack tip can be obtained using the COD results.

The structure of this paper is organized as follows: in Section 2, a brief review of the CBIE and HBIE formulations for 2-D crack problems is given. In Section 3, the main steps of fast multipole technique are briefly reviewed. More details can be found in [12]. In Section 4, several numerical examples of models with large number of cracks both in finite and infinite plates are simulated to show the feasibility of constant elements and the efficiency of the proposed approach. The paper concludes with some discussions and potential extensions in the Section 5.

2. Basic BIE formulations for multiple cracks analysis

Without loss of generality, consider a 2-D linear elastic solid Ω with boundary Γ (including the outer boundary and the surfaces of N cracks), where the body forces are neglected in this study. The displacements at the source point y can be expressed by the conventional BIE on the crack surface as follows [12–14]:

$$\gamma u_i(y) = \int_\Gamma \tau_j(x) U_{ij}(x,y) d\Gamma(x) - \int_\Gamma u_j(x) T_{ij}(x,y) d\Gamma(x),$$
$$y \in (\Gamma \cup \Omega) \tag{1}$$

where x is the field point; i and j denote the Cartesian components; u_i and t_i are the displacement and traction components at the field point x, respectively; $\gamma=1$ if $y \in \Omega$ and $\gamma=0.5$ if $y \in \Gamma$ where Γ is smooth. The two kernel functions $U_{ij}(x,y)$ and $T_{ij}(x,y)$ in Eq. (1) are respectively the Kelvin displacement and traction fundamental solutions which can be found in many BEM publications [1–3]. Once the components of relative displacement discontinuities on crack surface are known, the CODs will be given by Δu_i as follows:

$$\Delta u_i(x) = u_i(x)|_{x \in A^+} - u_i(x)|_{x \in A^-} \tag{2}$$

where A^+ and A^- represent the upper and the lower surfaces of the crack, respectively. Therefore, the displacement $u_i(y)$ can be determined by the traction $\tau_j(x)$ on the outer boundary Γ, the displacement $u_j(x)$ on the rest of the outer boundary. In BIE (1), integral with the U kernel is a weakly-singular integral, while the other one with the T kernel is a Cauchy principle-value (CPV) integral.

However, under the given boundary conditions, the solutions of the unknowns are still not sufficient to obtain by the sole CBIE of Eq. (1). Thus, to solve a given problem containing multiple cracks, an additional boundary integral equation should be constructed on the opposite surface of the crack, that is, the traction BIE. To do this, taking derivatives of Eq. (1) and applying the stress-strain relation, one can obtain the corresponding traction BIE as follows [12–14]:

$$\gamma \tau_i(y) = n_j(y) \int_\Gamma \tau_k(x) U_{ijk}(x,y) d\Gamma(x) - n_j(y) \int_\Gamma u_k(x) T_{ijk}(x,y) d\Gamma(x),$$
$$y \in (\Gamma \cup \Omega) \tag{3}$$

where n_j being the component of the outward unit directional cosine at the source point y. The two kernel functions $U_{ijk}(x,y)$ and $T_{ijk}(x,y)$ in Eq. (3) contain derivatives of $U_{ij}(x,y)$ and $T_{ij}(x,y)$ together with elastic constants. As the source and field points are coincided on the crack surfaces, the kernel functions $U_{ijk}(x,y)$ is a CPV integral and $T_{ijk}(x,y)$will exhibit a hypersingularity of order $O(1/r^2)$. Therefore, traction BIE (3) is also named hypersingular BIE (HBIE). The treatment of singular forms of traction integral Eq. (3), which contains strongly singular and hypersingular integrals, is very useful and also has been always a challenge for many BEM researchers. In the conventional BEM approach, the difficulty caused by hypersingular integrals was overcome by using the concept of the finite part integral. When constant elements are used, all the singular and hypersingular integrals in traction BIE (3) can be evaluated analytically which can be found in Ref. [12]. As a result, under the given displacement or traction boundary conditions, the unknowns can be solved uniquely by using the displacement and traction boundary integral equations of CBIE (1) and HBIE (3).

To solve CBIE (1) and HBIE (3) simultaneously, a dual BIE (CHBIE) formulation of using a linear combination of the CBIE and HBIE can be written as follows [12]:

$$\alpha \, \text{CBIE} + \beta \, \text{HBIE} = 0 \tag{4}$$

where α and β are the coupling constant. In this study, the constant variable α is set to be 1.0, while $\beta$is equal to the size of constant element length used in the model. More discussions on the choice of βcan be found in [15–18] for other cases. In these publications, the CHBIE (4) has been found to be very effective and efficient for solving potential, elastostatic, elastic wave and acoustic wave problems. This paper also shows that CHBIE (4) is efficient for crack problems.

With the displacements and tractions being first determined at the boundary and crack surfaces, the displacements, stress and strains at the any point in the domain and the crack tips can be computed. Once the entire unknown CODs on the crack surfaces are obtained by the fast multipole BEM, the stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ for each crack can be easily determined using one point COD formula at the crack-front elements as follows [2]:

$$K_{\rm I}=\frac{G}{(\kappa+1)}\sqrt{\frac{2\pi}{r}}\Delta u_n(r) \tag{5}$$

and

$$K_{\rm II}=\frac{G}{(\kappa+1)}\sqrt{\frac{2\pi}{r}}\Delta u_s(r) \tag{6}$$

where $K_{\rm I}$ and $K_{\rm II}$ are the SIFs of mode I and mode II at the crack tip; G is the shear modulus of materials; r is the radial distance from the observed point to the crack tip; $\Delta u_n(r)$ and $\Delta u_s(r)$ are the opening (normal) and sliding (tangential) components of the nodal displacements on the crack surfaces, respectively. The parameter $\kappa=3-4\nu$for plane strain, while $\kappa=(3-\nu)/(1+\nu)$ for plane stress, where ν is Poisson's ratio. It can be apparently seen that only the nodal displacements data near the crack tip are used to extract the SIFs according to Eqs. (5) and (6).

3. Solution procedures of the fast multipole BEM

The fast multipole techniques for solving potential, elastostatic, stokes flow and acoustic wave problems, i.e. using CHBIE (4) have been described in details in [5–12]. Since the kernel functions of a given crack problems are the same as in [12], the main steps of crack problems using the fast multipole BEM are summarized in this section. In this study, constant elements are applied to discretize the BIEs. All the moments are evaluated analytically, as well as in integrations of the kernel functions in the near-field direct evaluations. The details of the multipole and local expansions of the fast multipole BEM will not elaborated here since it is not main purpose of the present paper. As mentioned before, it is clear that if all the unknown CODs of a crack are obtained, the crack problems can be solved such as computing the stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ according to Eqs. (5) and (6). To solve multiple crack problems, the present solution procedure of the fast multipole BEM consists mainly of three stages, i.e., the initiation, the iteration and the post-process stages.

3.1. Initiation stage

The purpose of initiation stage is to prepare all the date related to the domain, constant elements, boundary conditions, maximum number of constant elements allowed in a leaf, the numbers of terms for multipole and local Taylor's expansion and GMRES solution convergence tolerance, etc. For a given multiple-crack problems, constant elements were applied to discretize the outer boundaries of the domain and crack surfaces in the same way as in conventional BEM.

3.2. Iteration stage

In this block, an iterative solver, GMRES is applied to solve the system equation and use the FMM to accelerate the multiplied vector in each step. Employing the GMRES iterative algorithm to solve the equation system can remarkably enhance the convergent rate compared with the conventional BEM because the entire matrix does not need to be stored in the memory. If the GMRES criterion is not satisfied or convergent, return to the initiation stage and check the crack model. Otherwise, if the criterion is satisfied, go to the next stage and then computing the corresponding stress intensity factors using Eq. (5) or Eq. (6).

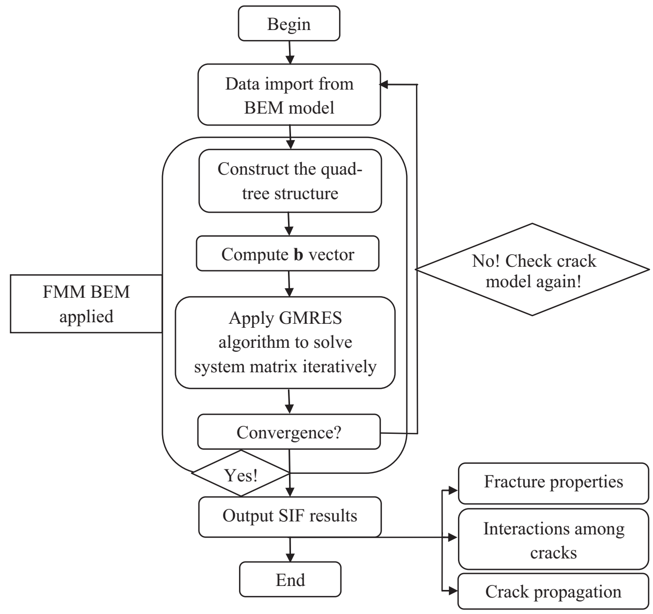

Fig. 1. The flow chart of present solution procedures.

3.3. Post-process stage

The post processes can be carried out according to the requirements or interests as follows:

(1) Comparing the COD results from the proposed approach and ANSYS software.
(2) Computing the fracture properties such as the maximum SIFs and compared with the fracture toughness of materials or critical stress intensity factor.
(3) Investigating the distribution of stress or strain fields near the crack tip.
(4) Considering the interactions among cracks when the cracks are much closed.
(5) Analyzing crack propagations once the stress intensity factors are correctly computed.

The solution procedures of the proposed approach are summarized in the flow chart as shown in Fig. 1.

4. Numerical examples

In this section, several numerical examples are presented to demonstrate the accuracy and efficiency of the fast multipole BEM for multiple crack problems. All the examples are chosen from those papers in the open literature. In all BEM models, the outer boundaries of the domain and crack surfaces are discretized by constant elements. The outer boundaries are discretized by 400 constant elements and each crack surface is discretized by 360 constant elements. The numbers of terms for both multipole and local expansions were set to 20, the maximum number of elements in a leaf to 100, and the tolerance for convergence of GMRES is chosen as 10^{-6}. The non-dimensional stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ of multiple crack problems in the finite and infinite domains can be solved and compared with those in literatures [19–23].

4.1. Stress intensity factors in finite plates

4.1.1. Rectangular plate with center crack

The first example is probably the simplest problem fracture mechanics, which is a rectangular plate contains a centered crack with various lengths constructed for the purpose of illustrating the computational procedure presented. The length of the crack is to allocated at the center of a rectangular plate of $2b$ width and $2h$ height. The width and the height of this plate are $2b=10$ mm and $2h=20$ mm, respectively. The material properties are $E=1$ Mpa and Poisson's ratio $\nu=0.3$. The loads applied at upper and lower edges are $\sigma=1$ Mpa as shown in Fig. 4. This example is analyzed by three approaches, i.e., the presented fast multipole BEM, ANSYS software and theoretical analysis. From the elastic theory, for a centered crack problem, the COD can be analytically expressed as follows:

$$\Delta u(x)=\frac{2(1-\nu)a}{G}\sigma^{\infty}\sqrt{1-\left(\frac{x}{a}\right)^2} \tag{8}$$

In ANSYS analysis, totally 4-node plane stress elements are used. The COD results on the crack surface predicted by the three numerical analyses, corresponding to the proposed approach, ANSYS software and elastic theory are compared as shown in Fig. 2. It can be seen from Fig. 2 that the fast multipole BEM results agree well with ANSYS and analytical solutions. The maximum error of fast multipole BEM and ANSYS is smaller than 2.0%, and 1.0% compared to analytical solutions, respectively.

It is important to point out that the relative errors of displacement fields near the crack tip are generally large even when some special crack tip elements are used. The behavior of the error distribution of COD fields should be known for a reliable numerical method. In general, the minimum error of COD at $x=0$ should be always less than 0.05 mm which can be evaluated since the analytical solutions are available for this simple problem. The minimum error of COD of this study is 0.03312 mm, meeting the requirement. The numerical results of COD error distributions which are calculated by the fast multipole BEM are compared as shown in Fig. 3, which illustrates the effectiveness of the proposed approach.

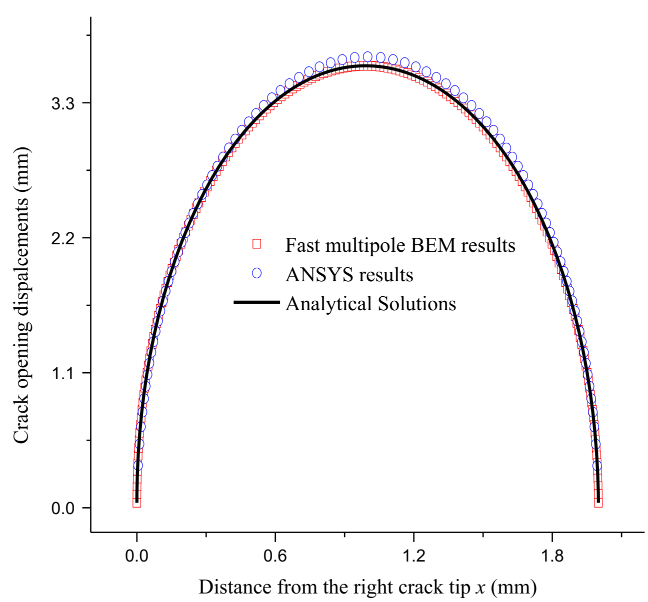

Fig. 2. Comparison of the CODs from present approach, ANSYS and elastic theory.

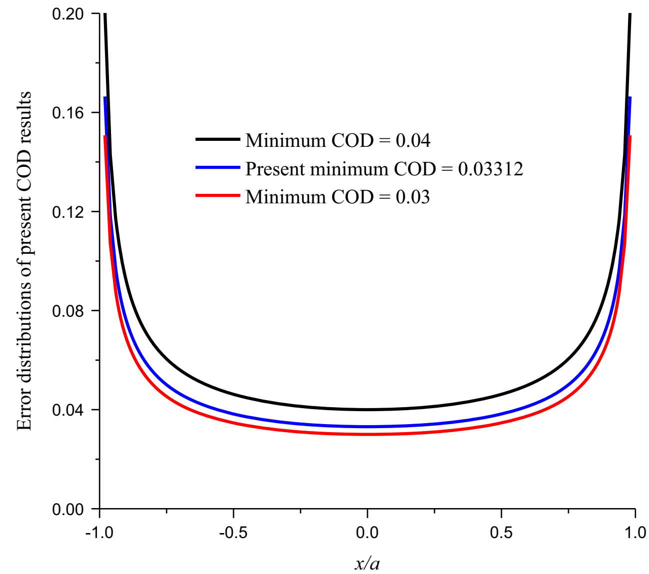

Fig. 3. Error distributions of the calculated COD fields with constant elements.

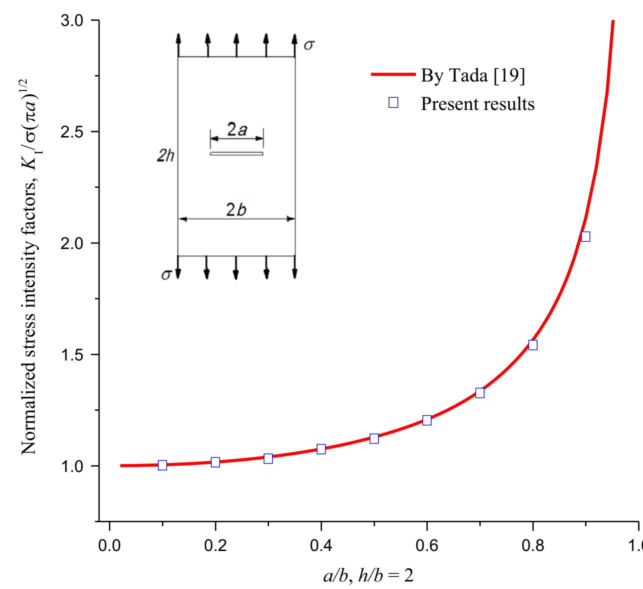

Fig. 4. Normalized stress intensity factors of center cracked finite rectangular plate.

The calculated results of normalized mode-I SIFs are compared with Tada [19] as shown in Fig. 4. A good agreement can be seen. And Fig. 5 shows the relative errors of these two approaches. We can find that the errors are relatively large when the crack tips are close to the boundary of the plate. Therefore, more elements are required near the crack tips and the influence of the plate width should be considered.

4.1.2. Square plate with four cracks

To show the capability of the proposed approach in solving the problem of arbitrary distributions of crack, a square plate with four cracks is analyzed. The plate is subjected to the uniformly distributed tension and the lengths of four cracks are same as shown in Fig. 6. This example was previously analyzed by Chen and Chen [20], in which used a special technique, that is, a dual boundary element method (D-BEM) to solve multi-crack problems [21]. In the proposed model, the outer boundaries are discretized

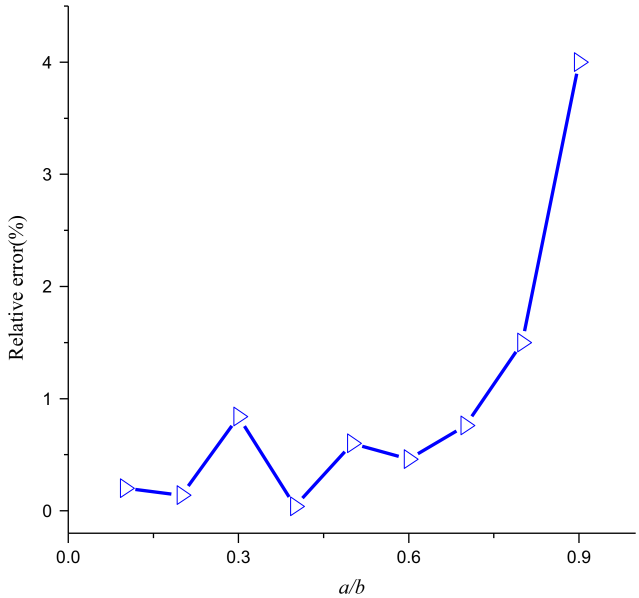

Fig. 5. Relative errors of the calculated SIFs.

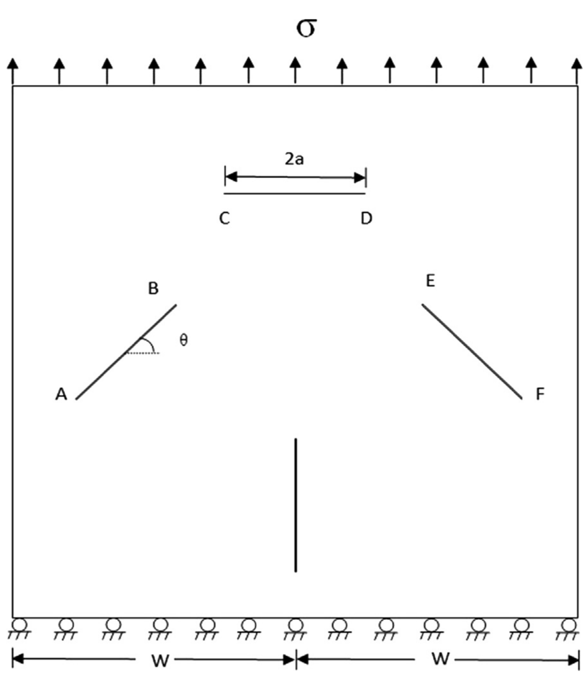

Fig. 6. A finite plates with four cracks.

Table 1
Non-dimensional SIFs for a square plate with four cracks.

θ (deg.)	*A, F*		*B, E*		*C, D*	
	Present	Ref. [20]	Present	Ref. [20]	Present	Ref. [20]
0	1.10322	1.1051	1.08252	1.0848	1.20355	1.2011
30	0.81924	0.8208	0.84031	0.8467	1.21412	1.2119
45	0.53120	0.5354	0.56544	0.5703	1.20193	1.1999
60	0.25287	0.2533	0.27289	0.2741	1.17886	1.771
90	0	0	0.00238	0.0024	1.13314	1.1329

by using 400 constant elements, while each crack surface is discretized by using 360 constant elements.

The results of the normalized stress intensity factors K_{I} of three cracks for the variation of angle θ are listed in Table 1. As mentioned in [20], the geometry and loading are symmetric in the perpendicular direction. So the SIFs of crack tips *A* and *F* should be the same, which is also true for crack tips *B* and *E*, and *C* and *D*. It can be seen from Table 1 that the computed results demonstrated the above conclusions, and agree well with the results in [20], which shows the accuracy of the proposed approach. More intuitive figure for crack tip *C* is shown in Fig. 7.

4.1.3. *Comparison of COD results for a square plate with 100 randomly distributed cracks*

In this section, we calculated a model of a 2-D square plate with 100 randomly distributed cracks as shown in Fig. 8. The total degree of freedoms is 36,400. In order to simplify this model, one crack is put at the center of the square plate as a research object (crack *A*), while other cracks are randomly distributed and any two cracks do not intersect. To obtain the CODs of crack *A*, a special region around the center crack is taken into consideration. The technique used here is similar to the author's previous work [22], all cracks are divided into two groups according to the non-dimensional radial distances of cracks to the current crack *A*, i.e. adjacent group and far-field group. The adjacent group contains those cracks with a relatively small circle but has strong interactions to the center crack, being placed in a circle in dashed-line around the crack *A*. The others, the cracks of far-field group are defined as those with relatively large distance located outside the

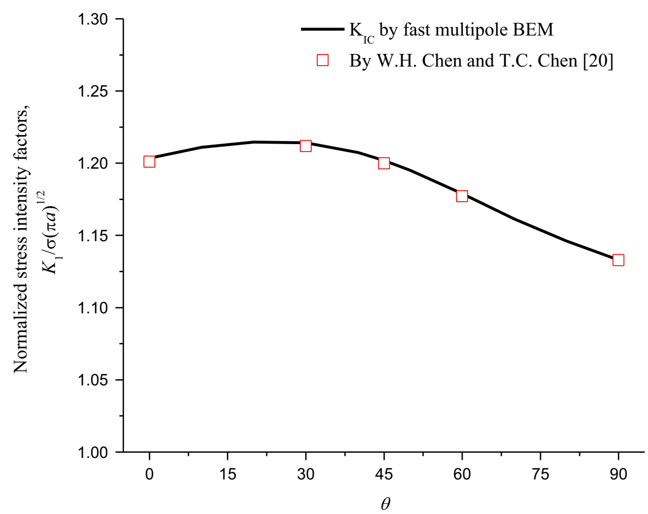

Fig. 7. Normalized stress intensity factors for the crack tip *C*.

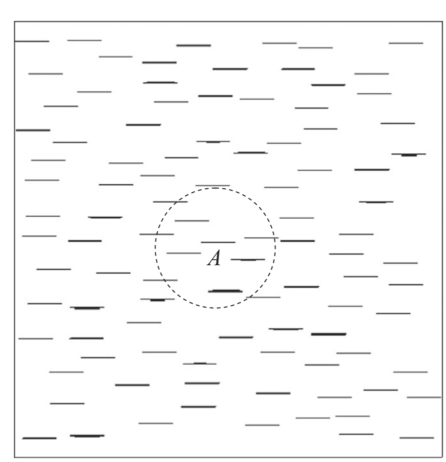

Fig. 8. 100 randomly distributed cracks in a square plate.

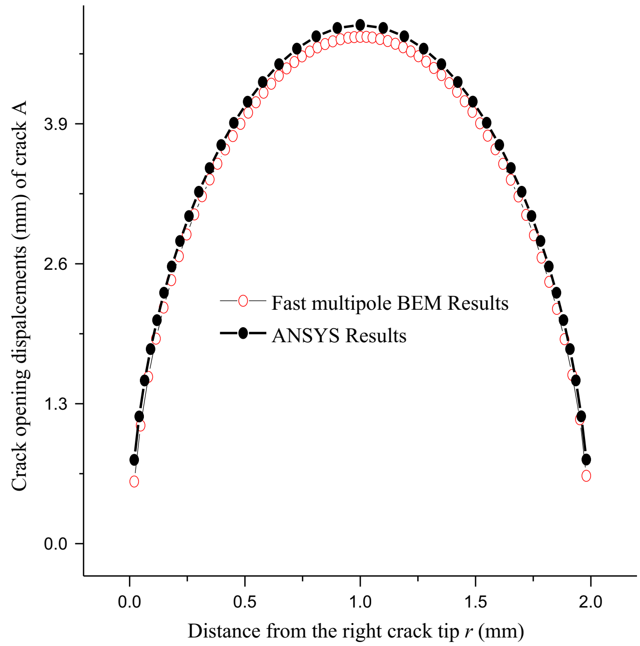

Fig. 9. Comparison of COD results for a finite plate with 100 random cracks.

dashed-line circle. In the ANSYS analysis, we neglect the interactions of far-field group since the interactions are relatively small. The CODs of the current crack *A*are calculated and compared by the fast multipole BEM and ANSYS as shown in Fig. 9. The total CPU time on a desk PC is within minutes. It can be seen from Fig. 9 that the COD results of the fast multipole BEM agree well with the results of ANSYS software, showing the fast multipole BEM with constant elements is effective and efficient in dealing with multiple random crack problems.

4.2. *Stress intensity factors in infinite plates*

4.2.1. *An infinite plate with three collinear cracks*

This example is an infinite plate in tension which contains three collinear cracks with equal length with 2160 DOFs. The computed results of normalized mode-I stress intensity factors are presented in Fig. 10. It can be seen that the numerical results from the proposed approach are in good agreement with the analytical solutions of Refs. [19].

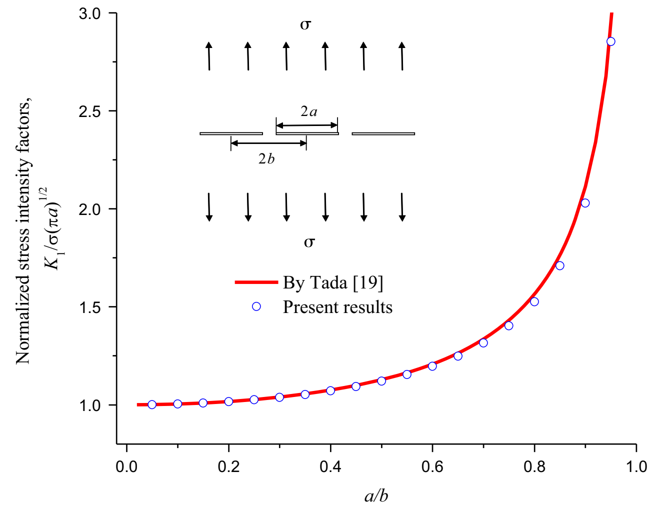

Fig. 10. Normalized stress intensity factors of three collinear cracks.

4.2.2. *One row of periodical collinear cracks in horizontal line*

The second example is one row of periodical collinear cracks in an infinite plate under a far-field tension perpendicular to the crack surfaces. The cracks are uniform distributed with equal size, same orientation and spacing as shown in Fig. 11. In this model, up to 501 cracks are taken into consideration instead of using an infinite number of cracks. Here the total numbers of cracks are set to be odd so that the middle crack can be centrally arranged along the *x* axis so and the post process is more convenient. The maximum total DOFs of this cracked model is over 180,000. The numerical results of normalized mode-I stress intensity factors are compared with Tada [19] as listed in Tables 2, and 3 shows the absolute value of errors of SIFs. We can see that the numerical results are gradually close to the analytical solutions and the absolute errors decrease as the number of crack increase. In general, the increase of the number of cracks can improve the accuracy of the present results. However, the CPU time and memory requirement will be long and large. At this point, an appropriate truncated number *N* can stand for this class of

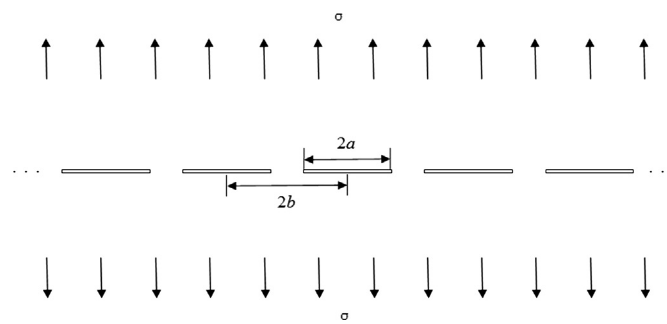

Fig. 11. Periodical collinear cracks with equal size under uniform tension.

periodic collinear cracks problem, saving the CPU time and memory requirement. In this manner, we can find an appropriate truncated number of cracks *N* to simulate periodic collinear cracks problem. It also can be seen that the absolute errors will exceed 1% when the ratio *a*/*b* is over 0.7. This is because the interactions among cracks will trend strong. Therefore, other technique can be used to reduce the errors when the space among cracks is much closed.

4.2.3. *One column of periodical cracks in vertical line*

The third example is a column of periodic cracks located vertically inside an infinite plate under a far-field uniform tension perpendicular to the crack surfaces. In this example, up to 101 cracks (total 36,360 DOFs). are taken into consideration instead of using an infinite number of cracks The normalized mode-I stress intensity factors are calculated by the proposed approach and compared with Chen [23] in Fig. 12, showing a good agreement between the present results and Chen's solutions. Correspondingly, it also shows the effectiveness and efficiency of the fast multipole BEM for solving this class of multiple cracks problem. According to the authors' best knowledge, the interactions among cracks which arranged in a vertical line are much stronger than those of cracks arranged in a collinear row of horizontal line. In this point, this class of crack problems should be studied in further when the cracks are much closed.

4.2.4. *Groups composed of two cracks with different lengths*

The fourth example is several groups of cracks placed periodically inside an infinite plate under far-field uniform load as shown in Fig. 13. Each group includes two cracks with unequal length: 2*a*

Table 2
The normalized mode-I SIFs as a function of *a*/*b* for one row of periodical collinear cracks.

N	*a*/*b*								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
11	1.00389	1.01598	1.03751	1.07094	1.12094	1.19621	1.31586	1.52926	2.03154
21	1.00401	1.01646	1.03864	1.07307	1.12457	1.20215	1.32555	1.54597	2.06681
31	1.00405	1.01662	1.03902	1.07381	1.12585	1.20421	1.32892	1.55181	2.07908
41	1.00407	1.01671	1.03922	1.07418	1.12647	1.20526	1.33063	1.55477	2.08535
51	1.00408	1.01674	1.03934	1.07441	1.12687	1.20589	1.33167	1.55657	2.08915
61	1.00409	1.01679	1.03942	1.07456	1.12712	1.20631	1.33236	1.55778	2.09174
71	1.00409	1.01681	1.03948	1.07467	1.12731	1.20662	1.33286	1.55865	2.09353
81	1.00411	1.01683	1.03952	1.07475	1.12744	1.20684	1.33323	1.55932	2.09491
91	1.00411	1.01685	1.03955	1.07481	1.12755	1.20702	1.33352	1.55979	2.09599
101	1.00411	1.01686	1.03958	1.07486	1.12765	1.20716	1.33376	1.56021	2.09685
201	1.00412	1.01691	1.03969	1.07509	1.12803	1.20781	1.33481	1.56205	2.10075
301	1.00412	1.01693	1.03974	1.07517	1.12816	1.20802	1.33517	1.56266	2.10206
401	1.00412	1.01694	1.03976	1.07521	1.12823	1.20813	1.33534	1.56297	2.10271
501	1.00412	1.01694	1.03977	1.07523	1.12827	1.20819	1.33545	1.56316	2.10311
Tada [19]	1.00415	1.01698	1.03983	1.07533	1.12838	1.20847	1.33601	1.56497	2.11331

Table 3
The absolute errors of SIFs as a function of *a*/*b* using different numbers of cracks *N*.

N	*a*/*b*								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
11	2.6E−4	1E−3	2.32E−3	4.39E−3	7.44E−3	1.25E−2	2.12E−2	3.57E−2	8.18E−2
21	1E−4	5.2E−4	1.19E−3	2.26E−3	3.81E−3	6.32E−3	1.05E−2	1.9E−2	4.65E−2
31	8E−5	3.6E−4	8.1E−4	1.52E−3	2.53E−3	4.26E−3	7.09E−3	1.32E−2	3.42E−2
41	8E−5	2.7E−4	6.1E−4	1.15E−3	1.91E−3	3.21E−3	5.38E−3	1.02E−2	2.8E−2
51	7E−5	2.4E−4	4.9E−4	9.2E−4	1.51E−3	2.58E−3	4.34E−3	8.4E−3	2.42E−2
61	6E−5	1.9E−4	4.1E−4	7.7E−4	1.26E−3	2.16E−3	3.65E−3	7.19E−3	2.16E−2
71	6E−5	1.7E−4	3.5E−4	6.6E−4	1.07E−3	1.85E−3	3.15E−3	6.32E−3	1.98E−2
81	4E−5	1.5E−4	3.1E−4	5.8E−4	9.4E−4	1.63E−3	2.78E−3	5.65E−3	1.84E−2
91	4E−5	1.3E−4	2.8E−4	5.2E−4	8.3E−4	1.45E−3	2.49E−3	5.18E−3	1.73E−2
101	4E−5	1.2E−4	2.5E−4	4.7E−4	7.3E−4	1.31E−3	2.25E−3	4.76E−3	1.65E−2
201	3E−5	7E−5	1.4E−4	2.4E−4	3.5E−4	6.6E−4	1.2E−3	2.92E−3	1.26E−2
301	3E−5	5E−5	9E−5	1.6E−4	2.2E−4	4.5E−4	8.4E−4	2.31E−3	1.13E−2
401	3E−5	4E−5	7E−5	1.2E−5	1.5E−4	3.4E−4	6.7E−4	2E−3	1.06E−2
501	3E−5	4E−5	6E−5	1E−5	1.1E−4	2.8E−4	5.6E−4	1.81E−3	1.02E−2

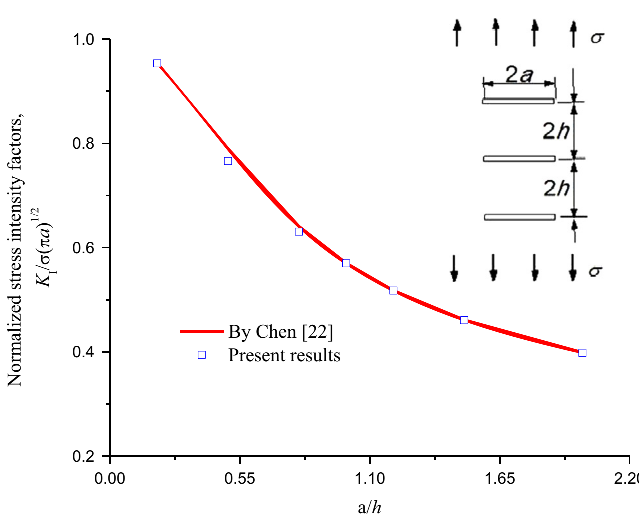

Fig. 12. Normalized stress intensity factors of column of periodical cracks.

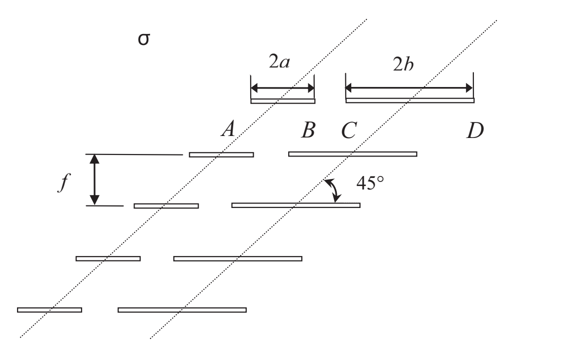

Fig. 13. Infinite groups composed of two cracks with unequal lengths.

and 2*b*, respectively. The horizontal distance between the two cracks of each group is 0.5*a*, while the vertical distance is a variable *f*. In this model, up to 13 groups are taken into consideration with total 18,720 DOFs. The modes I and II normalized stress intensity factors are respectively calculated by the proposed approach in comparison with those by Chen [23] as shown in Figs. 14 and 15, where a good agreement between the two approaches can be seen, showing the effectiveness of the fast multipole BEM in dealing with this kind of problem.

4.2.5. Double periodical collinear cracks

The last example considers an infinite plate containing double periodical collinear cracks with equal length under a far-field uniform tension. The number of rows is set to be an odd number *M* and the geometrical center of the middle row is located at the

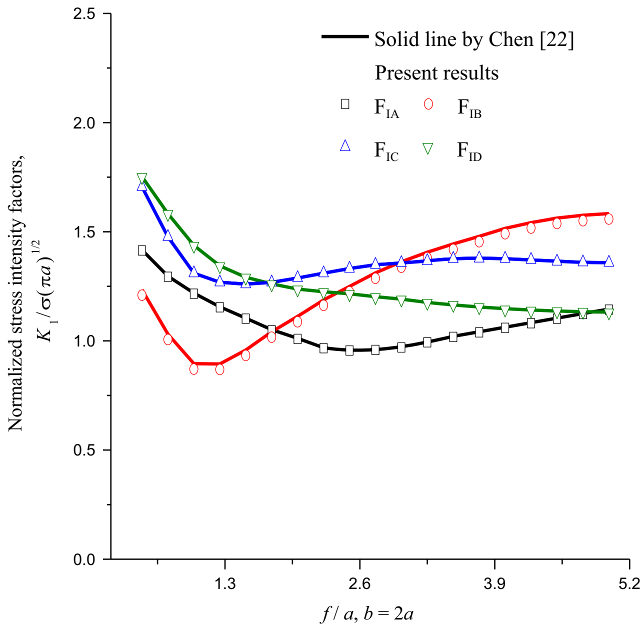

Fig. 14. The mode I normalized SIFs of two cracks with unequal lengths.

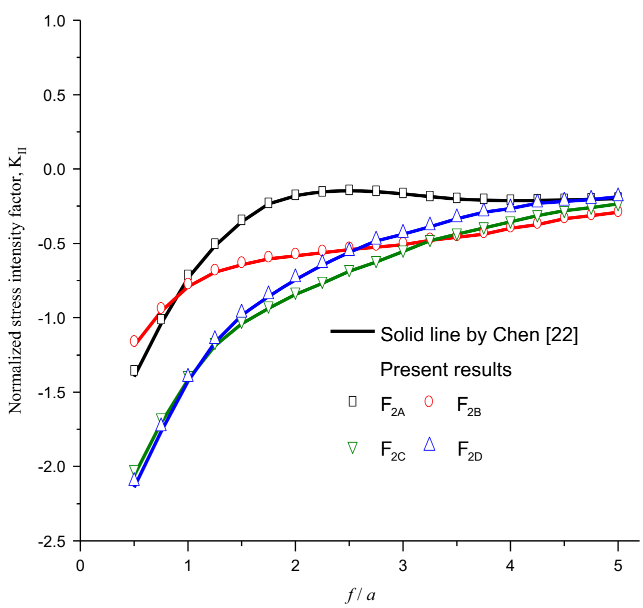

Fig. 15. The mode II normalized SIFs of Fig. 13.

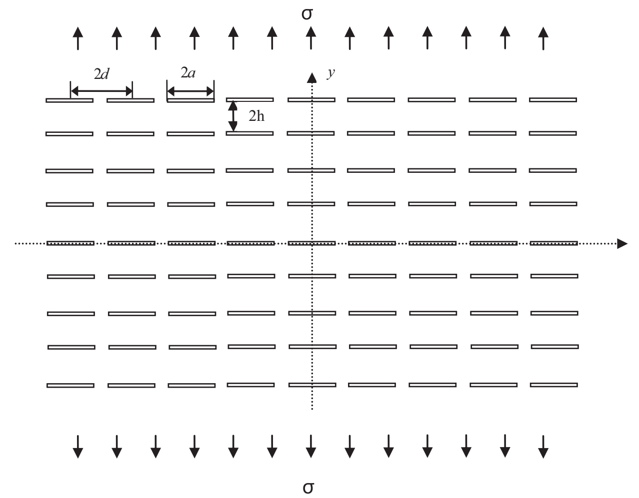

Fig. 16. Double periodical collinear cracks with equal length.

center of x axis as shown in Fig. 16. The total number of cracks is $M \times M$. The normalized stress intensity factors of the central crack at different number of rows M are calculated and compared with those by Wang and Feng [24] as listed in Table 4. The absolute errors of SIFs of some double periodical collinear cracks models are listed in Table 5. It can be seen that the numerical results agree well with analytical results and all the absolute value of errors are less than 1%, which shows the effectiveness of the proposed approach in dealing with double periodical collinear cracks problem.

The total DOFs, GMRES iterations and memory requirements of these models are listed in Table 6, respectively. It is easy to find that with the increase of crack numbers, the GMRES iterations increase steady. Also, the CPU time of different models, using the desk-top computer Dell with Inter Core Dual processor, 4 gigabytes of memory, 500 gigabytes hard drive and 2.4 GHz, are compared with Dual-BEM (D-BEM) as listed in Table 7. It can be seen the CPU time of the fast multipole BEM is larger than that of D-BEM when the numbers of cracks are relatively small. However, the CPU times of D-BEM will increase more quickly compared with fast multipole BEM when the total number of cracks grows. According to the author's knowledge, the calculation of fast multipole BEM has a linear relationship with the DOF of the original problems, but the slope of this linear relationship is large, which leading to the

Table 4
The normalized SIFs of double periodical collinear cracks at different rows.

Total number of cracks N	$h/d=0.5$					
	Present			Chen and Lin [23]		
	$a/d=0.5$	0.8	0.99	$a/d=0.5$	0.8	0.99
1×101	1.12769	1.56025	6.37012	1.128379	1.564972	6.370240
7×7	0.95743	1.47969	6.36438	0.962046	1.480109	6.368981
9×9	0.96203	1.47788	6.36445	0.962046	1.480106	6.368907
11×11	0.96204	1.47785	6.36437	0.962046	1.480104	6.368830
13×13	0.96204	1.47783	6.36431	0.962046	1.480101	6.368752

Table 5
Absolute errors of stress intensity factors.

Total number of cracks N	a/d		
	0.5	0.8	0.99
1×101	6.0E−4	4.72E−3	1.2E−4
7×7	4.8E−3	4.19E−4	4.60E−3
9×9	1.7E−4	2.23E−3	4.457E−3
11×11	1.7E−4	2.25E−3	4.46E−3
13×13	1.7E−4	2.27E−3	4.44E−3

Table 6
DOFs, iterations and memory requirements of the fast multipole BEM.

Total number of cracks N	$h/d=0.5$, $a/d=0.5$		
	DOFs	GMRES iterations	Memory (Mb)
7×7	35,280	8	88.8
9×9	58,320	9	90.0
11×11	87,120	9	91.5
13×13	121,680	9	93.4

Table 7
Comparison of CPU times of the two approaches.

Total number of cracks N	CPU time (s)	
	Fast multipole BEM	D-BEM
7×7	189.182	62.790
9×9	326.978	227.449
11×11	469.282	667.497
13×13	666.405	1690.458

advantage of the fast multipole BEM only can truly reflected in large-scal multi-crack problems. It can be potentially concluded that the fast multipole BEM is more efficient than D-BEM in dealing with large-scale crack problems.

5. Conclusions and further extensions

A fast multipole BEM was applied to solve 2-D multiple crack problems. A symmetric dual BIE formulation was employed and constant elements were used to discretize the outer boundaries of the domain and crack surfaces. There is no need to use any other special crack-tip elements such as singular quarter-point elements in the discretization near the crack tips. The effectiveness and efficiency of the proposed approach were verified by computing the SIFs in several numerical examples. Numerical results show that the fast multipole BEM with constant elements is effective and efficient for analyzing multiple cracks in both finite and infinite domains.

Potential extensions include studies of the interactions among cracks when they are much closer to each other, and crack propagations based on the calculated stress intensity factors can be considered. The 2-D work with the fast multipole BEM can also be extended to cracks of various shapes and in 3-D models. Other fast solution methods for the BEM, such as the adaptive cross approximation method, can also be applied to study the multiple crack problems.

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