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A new boundary meshfree method with distributed sources

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ABSTRACT

A new boundary meshfree method, to be called the boundary distributed source (BDS) method, is presented in this paper that is truly meshfree and easy to implement. The method is based on the same concept in the well-known method of fundamental solutions (MFS). However, in the BDS method the source points and collocation points coincide and both are placed on the boundary of the problem domain directly, unlike the traditional MFS that requires a fictitious boundary for placing the source points. To remove the singularities of the fundamental solutions, the concentrated point sources can be replaced by distributed sources over areas (for 2D problems) or volumes (for 3D problems) covering the source points. For Dirichlet boundary conditions, all the coefficients (either diagonal or off-diagonal) in the systems of equations can be determined analytically, leading to very simple implementation for this method. Methods to determine the diagonal coefficients for Neumann boundary conditions are discussed. Examples for 2D potential problems are presented to demonstrate the feasibility and accuracy of this new meshfree boundary-node method.

1. Introduction

The method of fundamental solutions (MFS) has been studied for many years along with the boundary element and other boundary methods [1]. The MFS uses only the fundamental solution, which is the response due to a concentrated point source, in the construction of the solution of a problem without using any integrals. It is a natural boundary meshfree method and offers several advantages as compared with the BEM. First, meshing a boundary with only nodes is certainly much easier than with elements. Second, singular integrals are avoided in the MFS (although singularities of the kernel still play an important role). Third, programming with the conventional MFS is significantly simplified as compared with the BEM. All these advantages with the MFS have attracted continued interests from researchers. Comprehensive reviews on the MFS for various applications can be found in Refs. [2–4]. Some work on the MFS can be found in Refs. [5–10] for potential and elastostatic problems.

In the traditional MFS, a fictitious boundary slightly outside the problem domain is required in order to place the source points and avoid the singularity of fundamental solutions. The determination of the distance between the real boundary and the fictitious boundary is based on experience and therefore troublesome. In recent years, various efforts have been made aiming to remove this barrier in the MFS, so that the source points can be placed on the real boundary directly. Young et al. [11,13] and Chen et al. [12] proposed to place the source points on the boundary in the MFS and novel ways to determine the diagonal coefficients directly for simple geometries or using the results from the BEM based on the fact that the MFS and the indirect boundary integral formulation are similar in nature. In their approach, information of the neighboring points before and after each source point is needed in general in order to form line segments for integrating the kernels to obtain the diagonal coefficients. This is essentially the same information of the element connectivity in a BEM mesh. Šarler [14] proposed a similar modified MFS, where the diagonal terms are determined by the integration of the fundamental solution on line segments formed by using neighboring points, and the use of a constant solution to determine the diagonal coefficients from the derivatives of the fundamental solution. This approach is very stable, as is also shown in this study, but it amounts to solving the problems twice and is therefore not amenable to the fast multipole method using iterative solvers for the MFS [10,15]. Chen and Wang [16] recently proposed a similar method for determining the diagonal coefficients in the modified MFS by applying a known solution inside the domain, so that the diagonal coefficients from both the fundamental solution and its derivative can be determined indirectly, without using any element or integration concept. Again, this approach is appealing, stable, and accurate but is costly for solving large-scale problems due to the need to solve the problem twice.

In the spirit of pursuing truly meshfree boundary methods, we present in this paper a new boundary meshfree approach based

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on the modified MFS that has no fictitious boundaries and singularities. In this new approach, to be called *boundary distributed source* (BDS) method, the concentrated point sources are replaced with area-distributed sources covering the source points for 2D problems. These area-distributed sources are analytical integration of the original singular fundamental solution and its derivative so that they preserve the advantage of diagonal dominance for the system of equations, while they have no troublesome singularity issues. This BDS approach also does not require the information about the neighboring points for each source point, thus is a truly meshfree boundary method. Implementation of the method is easy and extension to 3D is straightforward where volume-distributed sources covering the source points can be applied. Although there are remaining issues with this BDS method, it is a very promising boundary meshfree method because it can be accelerated readily with the fast multipole method [10,15] or other fast solution methods in solving large-scale engineering problems.

2. Formulation of the boundary distributed source method

We consider the following Laplace equation governing potential problems in a 2D domain V (Fig. 1):

$$\nabla^2 \phi(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in V \tag{1}$$

under the following boundary conditions (BCs):

$$\phi(\mathbf{x}) = \overline{\phi}(\mathbf{x}), \quad \forall \mathbf{x} \in S_\phi \text{ (Dirichlet BC)} \tag{2}$$

$$q(\mathbf{x}) \equiv \frac{\partial \phi}{\partial n}(\mathbf{x}) = \overline{q}(\mathbf{x}), \quad \forall \mathbf{x} \in S_q \text{ (Neumann BC)} \tag{3}$$

where ϕ is the potential field, $S = S_\phi \cup S_q$ the boundary of V, n the outward normal, and the barred quantities indicate the given values on the boundary.

Let us place N distributed sources at point $\mathbf{y}_j$ ($j=1, 2, \ldots, N$) on boundary S (Fig. 1). We can show that ϕ given by the following expression satisfies the governing Eq. (1):

$$\phi(\mathbf{x}) = \sum_{j=1}^{N} \int_{A(\mathbf{y}_j)} G(\mathbf{x},\mathbf{y}') dA(\mathbf{y}') \mu_j, \quad \forall \mathbf{x} \in V - \bigcup_{j=1}^{N} A(\mathbf{y}_j) \tag{4}$$

where $A(\mathbf{y}_j)$ can be a line segment or an area covering point $\mathbf{y}_j$ on the boundary,

$$G(\mathbf{x},\mathbf{y}') = \frac{1}{2\pi} \log\left(\frac{1}{r}\right) \tag{5}$$

is the fundamental solution for 2D potential problems with r being the distance between the collocation point $\mathbf{x}$ and source point $\mathbf{y}'$, and μ_j an unknown intensity of the distributed source at $\mathbf{y}_j$. Note that $A(\mathbf{y}_j)$ can be a straight line segment in the tangential or normal direction of the boundary at $\mathbf{y}_j$, a ring around $\mathbf{y}_j$, or an area covering $\mathbf{y}_j$. If $A(\mathbf{y}_j)$ is a line segment in the tangential direction of the boundary and formed by the two mid-points between $\mathbf{y}_j$ and the points before and after point $\mathbf{y}_j$, then expression (4) is simply the indirect boundary integral equation (BIE) formulation based on the single-layer potential given in Eq. (5). Among all those choices, it was found that the area-distributed sources for 2D problems yield the most stable and accurate approach.

Therefore, in this paper, we consider the case that $A(\mathbf{y})$ is a circular disk of radius R and centered at point $\mathbf{y}$ on the boundary S (Fig. 2) for 2D problems. It can be shown that, for example, using direct integration in polar coordinates (see, e.g., Ref. [17], p. 108, Eq. (7.19)), the integration of the fundamental solution $G(\mathbf{x},\mathbf{y})$ on a circular disk $A(\mathbf{y})$ yields the following analytical results (Fig. 2):

$$\tilde{G}(\mathbf{x},\mathbf{y}) \equiv \int_{A(\mathbf{y})} G(\mathbf{x},\mathbf{y}') dA(\mathbf{y}') = \begin{cases} \dfrac{R^2}{2} \log\left(\dfrac{1}{a}\right) & \text{for } a > R; \\ \dfrac{R^2}{2} \log\left(\dfrac{1}{R}\right) + \dfrac{R^2 - a^2}{4} & \text{for } a \le R; \end{cases} \tag{6}$$

in which a is the distance between $\mathbf{x}$ and $\mathbf{y}$ (center of the disk, see Fig. 2). Notice that $\tilde{G}(\mathbf{x},\mathbf{y})$ is continuous when $\mathbf{x}$ crosses the edge of the disk (that is, at $a=R$). In comparison with the *concentrated point sources*, like the fundamental solution $G(\mathbf{x},\mathbf{y})$, we can call $\tilde{G}(\mathbf{x},\mathbf{y})$ given in Eq. (6) a *distributed area source*. Note also the relation between $\tilde{G}(\mathbf{x},\mathbf{y})$ and $G(\mathbf{x},\mathbf{y})$ in the limit: $\tilde{G}(\mathbf{x},\mathbf{y})/A(\mathbf{y}) = \tilde{G}(\mathbf{x},\mathbf{y})/\pi R^2 \to G(\mathbf{x},\mathbf{y})$, as $R \to 0$, using the first expression in Eq. (6).

Applying the new notation $\tilde{G}(\mathbf{x},\mathbf{y})$, we can rewrite Eq. (4) as follows:

$$\phi(\mathbf{x}) = \sum_{j=1}^{N} \tilde{G}(\mathbf{x},\mathbf{y}_j) \mu_j, \quad \forall \mathbf{x} \in V - \bigcup_{j=1}^{N} A(\mathbf{y}_j) \tag{7}$$

with $\tilde{G}(\mathbf{x},\mathbf{y})$ being given by Eq. (6). This is an expression of the solution for ϕ inside the domain except for all the disks, once the unknown densities μ_j are found on boundary S.

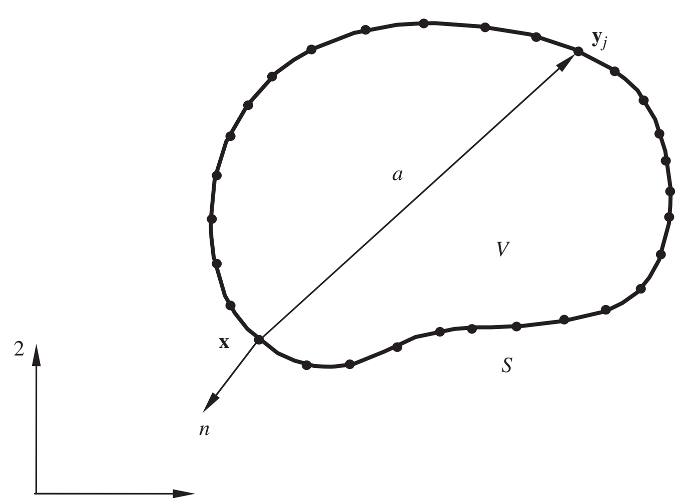

Fig. 1. A domain V with boundary S, the collocation point $\mathbf{x}$, and center $\mathbf{y}_j$ of a source.

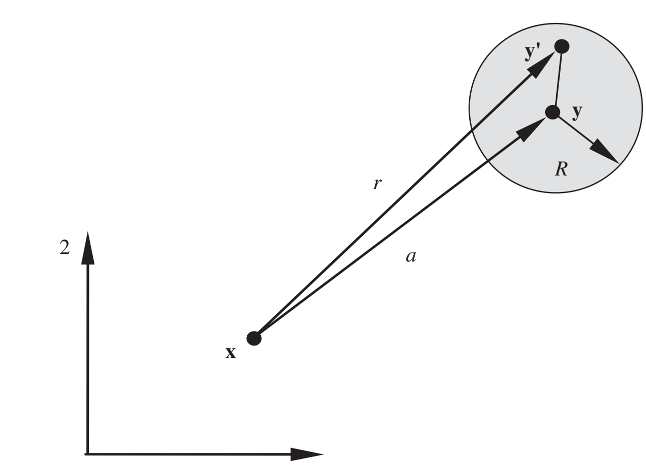

Fig. 2. Distributed source on a circular disk $A(\mathbf{y})$ centered at point $\mathbf{y}$ and with radius R.

The boundary conditions in Eqs. (2) and (3) can be satisfied at the collocation points $\mathbf{x}_i$ by adjusting μ_j at the source points $\mathbf{y}_j$, that is, imposing the following conditions:

$$\sum_{j=1}^{N} \tilde{G}_{ij}\mu_j = \overline{\phi_i} \quad \text{for } \mathbf{x}_i \in S_\phi \tag{8}$$

or

$$\sum_{j=1}^{N} \tilde{K}_{ij}\mu_j = \overline{q_i} \quad \text{for } \mathbf{x}_i \in S_q \tag{9}$$

where $\tilde{G}_{ij} \equiv \tilde{G}(\mathbf{x}_i,\mathbf{y}_j)$ and is determined by Eq. (6), and $\tilde{K}_{ij} \equiv \tilde{K}(\mathbf{x}_i,\mathbf{y}_j)$ with

$$\tilde{K}(\mathbf{x},\mathbf{y}) \equiv \int_{A(\mathbf{y})} K(\mathbf{x},\mathbf{y}')dA(\mathbf{y}') = \int_{A(\mathbf{y})} \frac{\partial G(\mathbf{x},\mathbf{y}')}{\partial n(\mathbf{x})} dA(\mathbf{y}') = \frac{\partial \tilde{G}(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}$$
$$= -\frac{R^2}{2a}\frac{\partial a}{\partial n(\mathbf{x})}, \quad \text{for } a > R \tag{10}$$

Currently, valid expressions for $\tilde{K}(\mathbf{x},\mathbf{y})$ when $0 \le a \le R$ are still under investigation. Therefore, the diagonal term in Eq. (9) needs to be determined indirectly for collocation points on S_q (the part of the boundary with Neumann boundary conditions). In this paper, the method proposed by Šarler [14] is applied to determine the diagonal coefficient in Eq. (9). In this approach, we first assume a constant solution, e.g., $\phi(\mathbf{x})=c$ everywhere. Then, from Eq. (8) we can solve for the corresponding densities μ_j^c for all the boundary points. Finally, from Eq. (9) we arrive at the following expression for the diagonal term using the known density values μ_j^c:

$$\tilde{K}_{ii} = -\frac{1}{\mu_i^c}\sum_{\substack{j=1 \\ j\neq i}}^{N} \tilde{K}_{ij}\mu_j^c \tag{11}$$

This technique is similar to the one used in the BEM for determining the diagonal coefficients by applying a rigid-body motion or the identities for the fundamental solutions when the singular integrals cannot be determined readily [15,18].

The following standard linear system of equations is formed after applying either Eqs. (8) or (9) at all the collocation points $\mathbf{x}_i$ (i=1, 2,…,N):

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{Bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix} \quad \text{or } \mathbf{A\mu} = \mathbf{b} \tag{12}$$

where $\mathbf{A}$ is the coefficient matrix, $\boldsymbol{\mu}$ the unknown density vector, and $\mathbf{b}$ the right-hand side vector. Once all the values of μ_j are determined by solving this equation, the potential at any point inside the domain or on the boundary can be evaluated using Eq. (7).

3. Examples

Examples for solving 2D potential problems are presented in this section to show the feasibility and potentials of the proposed boundary distributed source method and its comparison with the traditional method of fundamental solutions.

3.1. A square domain with Dirichlet BC

A square domain covering $0 \le x,y \le 1$ is studied first, which is the example used in Ref. [16]. Dirichlet BC is imposed on the four edges of this domain using the following analytical solution:

$$\phi(x,y) = x^2 - y^2 \tag{13}$$

The number of boundary nodes used is from 100 to 4000. A plot of the distributed sources with the circular disks is shown in Fig. 3 for the case with 100 nodes. The radius of the circular disk for the distributed area source covering each node is set as $R=d/2$, where d is the smallest distance between two nodes on the boundary. A number of M=101 field points are selected inside the domain along the line y=0.5 with $0.001 \le x \le 0.999$, and the solutions at these field points are computed and compared with the analytical solution. The relative error of the numerical solution is defined as

$$\text{Err} = \left[\frac{1}{M}\sum_{k=1}^{M}(|\phi_k - \overline{\phi}_k|/|\phi_k|)^2\right]^{1/2} \tag{14}$$

where ϕ_k and $\overline{\phi}_k$ are the analytical (Eq. (13)) and numerical solutions, respectively, at the k-th field point (which is excluded from the error calculation, if ϕ_k=0). A version of the traditional method of fundamental solution [10] is also used for solving this problem. For consistency, the distance of the fictitious boundary from the true boundary for the MFS is set as R as defined above for the boundary distributed source method.

Fig. 4 shows the relative errors in the results obtained using the BDS method (BDS Results) and the traditional MFS (MFS Results). Both BDS and MFS results converge with the errors below 1E−3 when the numbers of boundary nodes approach above 1000. However, the MFS results are one order of magnitude more accurate than the BDS results in this case. It was also found that the results of the BDS method are not sensitive to the value of the radius R of the circular disks used, although a general conclusion cannot be drawn at present. The values of R in the range $0 < R \le 0.6d$ were found to be valid, that is, the disks on the boundary can even overlap each other.

3.2. A circular domain with Dirichlet BC

A circular domain of radius=2 m is studied next, which is an example used in Ref. [13]. Dirichlet BC is imposed on the edge of the circle using the following analytical solution:

$$\phi(r,\theta) = r^6\cos(6\theta) \tag{15}$$

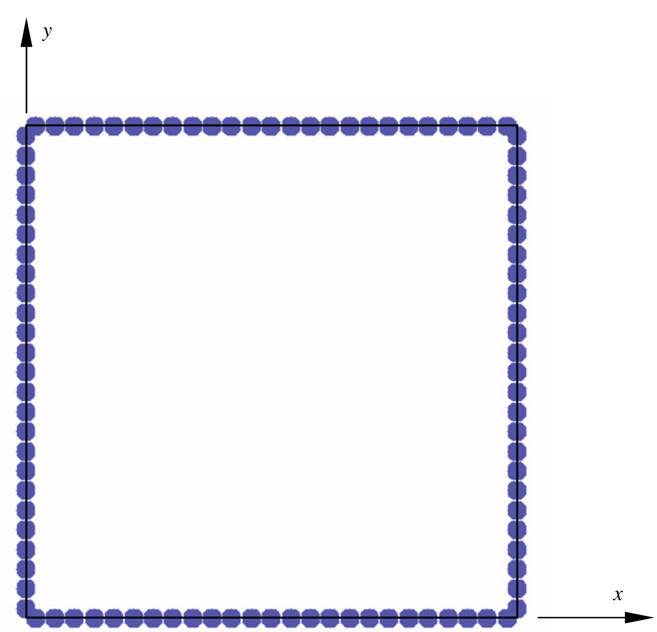

Fig. 3. A square domain with the boundary covered with 100 circular disks.

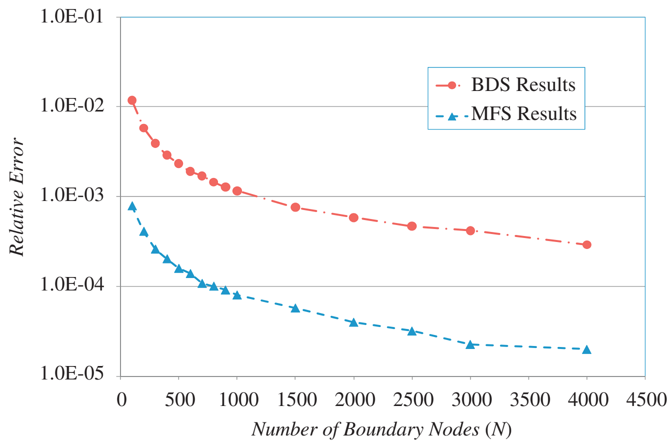

Fig. 4. Relative errors in the results for the square domain model with Dirichlet BC.

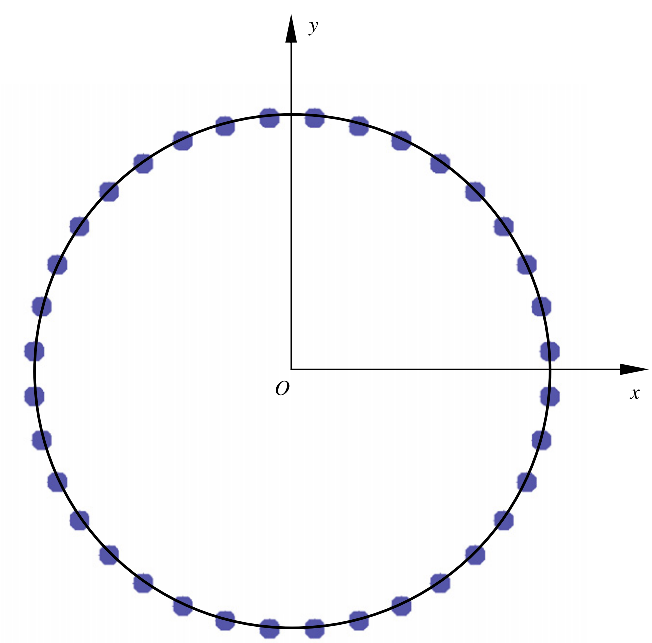

Fig. 5. A circular domain with the boundary covered with 36 circular disks.

in the polar coordinate (r,θ). The number of boundary nodes used are from 36 to 1200. A plot of the distributed sources with the circular disks is shown in Fig. 5 for the circle with 36 nodes. The radius of the circular disk for the distributed area source covering each node is set as $R=d/4$ in this case, with d being the smallest distance between two nodes. A total number of 120 field points are selected inside the circle along the line $r=1$ m and with $0^\circ \le \theta \le 360^\circ$, and the solutions at these field points are computed and compared with the analytical solution (Eq. (15)). The traditional MFS [10] is also used for comparison. Again, for consistency the distance of the fictitious boundary from the true boundary for the MFS is set as R, as used for the BDS method.

Fig. 6 shows the relative errors in the computed potential at the field points inside the domain using the BDS method and compared with the results using the MFS. In this example, the BDS results have better accuracy than the MFS results. Plots of the computed potentials at the 120 field points using the BDS method with 36 and 360 nodes on the boundary are given in Fig. 7. Excellent agreements of the BDS results with the analytical solution are observed.

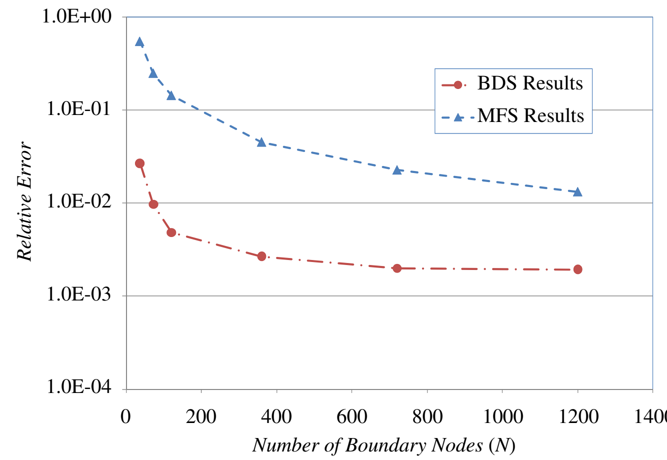

Fig. 6. Relative errors in the results for the circular domain model with Dirichlet BC.

3.3. A square domain with mixed BCs

Finally, we consider the square domain covering the region $0 \le x,y \le 1$ studied in the first example (Fig. 3), but with mixed boundary conditions to verify the formulas for computing the coefficients $\tilde{K}_{ij}$ given in Eqs. (10) and (11). The considered BCs are: along the edges $y=0$ and 1: $q=0$; along the edge $x=0$: $\phi=0$; and along the edge $x=1$: $\phi=100$. The analytical solution for this problem is

$$\phi(x,y) = 100x \quad (16)$$

Again, the number of boundary nodes used are from 100 to 4000 and the radius of the circular disk for the distributed area source is set as $R=d/2$, as in the first example. The same number of $M=101$ field points are placed inside the domain along the line $y=0.5$ with $0.001 \le x \le 0.999$, and the solutions of ϕ on both the boundary and at these field points are computed and compared with the analytical solution.

Fig. 8 shows the relative errors in the computed results at the nodes on the boundary and at the field points inside the domain using the BDS method. The boundary solutions converge smoothly with the increase of the number of the boundary nodes, indicating that the proposed BDS method works well with mixed BCs. In this case, on the part of the boundary with the Neumann BC, Eq. (10) is applied for calculating the off-diagonal coefficients $\tilde{K}_{ij'}$ and Eq. (11) is applied for calculating the diagonal coefficients $\tilde{K}_{ii}$. The use of Eq. (11) for determining the diagonal terms for problems with Neumann BC will require solving the problem twice, thus doubling the solution time in these cases. Fig. 8 also shows that the solutions at the field points inside the domain are less accurate than those in the first example where only Dirichlet BC is applied and Eq. (11) is not used. Therefore, the use of Eq. (11) may also add additional errors in the solutions. It is desirable to find the analytical expression for the diagonal term $\tilde{K}_{ii}$ for solving problems with Neumann BC in order to improve both the accuracy and efficiency of the solutions using the proposed BDS method.

4. Discussions

A new boundary meshfree method, termed boundary distributed source (BDS) method, is proposed in this paper for solving boundary-value problems with the nodes on the boundary of the problem domain only. The method is similar to the modified

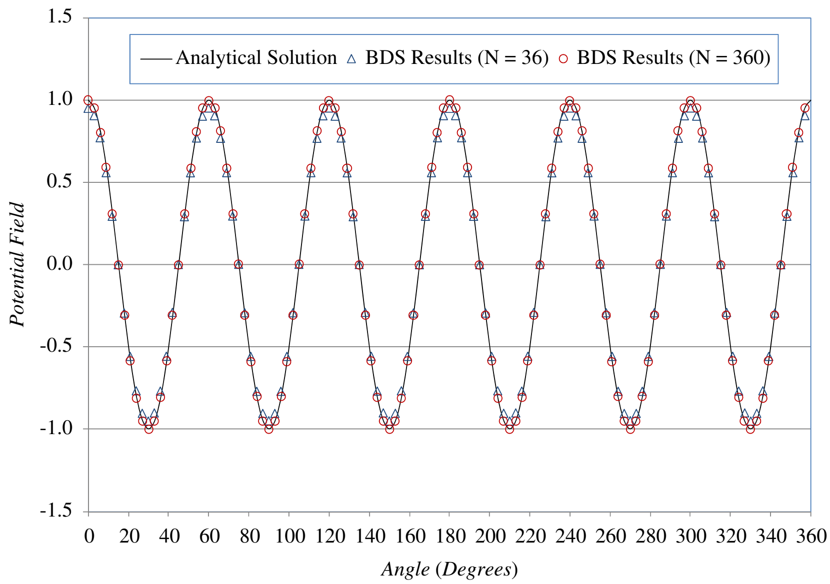

Fig. 7. Computed potential values at the field points for the circular domain model.

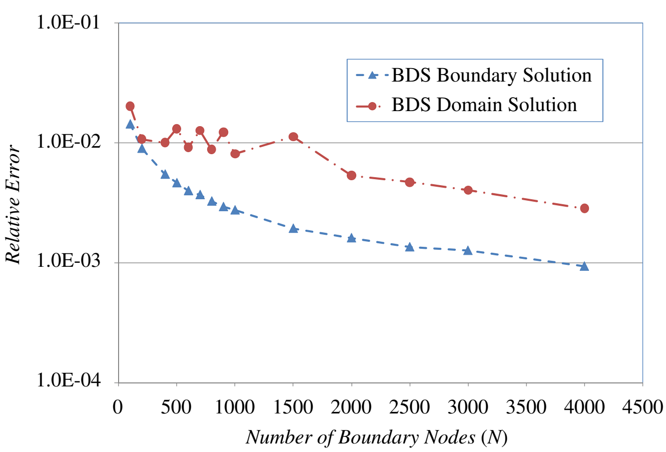

Fig. 8. Relative errors in the results for the square domain model with mixed BCs.

method of fundamental solutions in that both methods apply the source points on the boundary directly so that they are coincident with the collocation points. However, in the BDS approach, the singular fundamental solution is integrated first over small areas (for 2D) or volumes (for 3D) covering the source points so that the coefficients in the system of equations can be evaluated analytically and consistently, leading to extremely simple computer implementation for this method. Numerical examples using 2D potential problems clearly demonstrate the feasibility and accuracy of this BDS approach. Extension of this approach to 3D potential, 2D/3D elasticity, and other problems is straightforward.

There are a few remaining questions with the proposed BDS method. First, analytical expression for $\tilde{K}(\mathbf{x},\mathbf{y})$ shown in Eq. (10) when $0 \le a \le R$ has not been established. Therefore, the diagonal coefficients for equations on the boundary with Neumann BC have to be determined indirectly using Eq. (11), which reduces the computational efficiency and hinders large-scale applications of the BDS method. It is well-known from the BEM that the integration of the K kernel on the boundary, which is the normal derivative of the G kernel with respect to the collocation point $\mathbf{x}$, has a jump term when $\mathbf{x}$ approaches the boundary. This jump term definitely will have an effect on the area integral (for 2D) as defined for $\tilde{K}(\mathbf{x},\mathbf{y})$ in Eq. (10), as $\mathbf{x}$ approaches the area (see Fig. 2). The property of the jump term in $\tilde{K}(\mathbf{x},\mathbf{y})$ needs to be investigated carefully in order to obtain a valid expression for $\tilde{K}(\mathbf{x},\mathbf{y})$ when $0 \le a \le R$.

In addition, the expression in Eq. (4) does not satisfy the governing equation in the boundary regions where the disks overlap with the domain V (i.e., in regions $V \cap [\cup_{j=1}^{N} A(\mathbf{y}_j)]$), which can lead to inaccurate results near the disks. The proposed BDS method also seems to be indifferent to the differences between the interior or exterior domain problems, because only nodes on the boundary are required in the discretization. Thus it may cause problems or need special treatment if exterior domain problems [12] are to be solved. All the references to the boundary direction, outward normal, and so on, which are indicative of the domain type of the problem, are not present in the current BDS solution procedure. There are many options in improving the BDS formulation, for example, by moving the distributed sources outside the boundary of the domain, using areas of different shapes for the distributed sources or applying a different kernel function instead of the G kernel. These improvements and extensions to solving other types of problems need to be explored in the near future.

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References

[1] Mukherjee S, Mukherjee YX. Boundary methods: elements, contours, and nodes.. Boca Raton: CRC; 2005.
[2] Fairweather G, Karageorghis A. The method of fundamental solutions for elliptic boundary value problems. Advances in Computational Mathematics 1998;9(1/2):69–95.
[3] Golberg MA, Chen CS. The method of fundamental solutions for potential, Helmhotz and diffusion problems. In: Golberg MA, editor. Boundary integral methods: numerical and mathematical aspects. Boston: Computational Mechanics Publications; 1998. p. 103–76.
[4] Fairweather G, Karageorghis A, Martin PA. The method of fundamental solutions for scattering and radiation problems. Engineering Analysis with Boundary Elements 2003;27(7):759–69.

[5] Berger JR, Karageorghis A. The method of fundamental solutions for heat conduction in layered materials. International Journal for Numerical Methods in Engineering 1999;45(11):1681–94.
[6] Ramachandran PA. Method of fundamental solutions: singular value decomposition analysis. Communications in Numerical Methods in Engineering 2002;18(11):789–801.
[7] Smyrlis Y-S, Karageorghis A. A matrix decomposition MFS algorithm for axisymmetric potential problems. Engineering Analysis with Boundary Elements 2004;28(5):463–74.
[8] Mitic P, Rashed YF. Convergence and stability of the method of meshless fundamental solutions using an array of randomly distributed sources. Engineering Analysis with Boundary Elements 2004;28(2):143–53.
[9] Poullikkas A, Karageorghis A, Georgiou G. The method of fundamental solutions for three-dimensional elastostatics problems. Computers and Structures 2002;80(3–4):365–70.
[10] Liu YJ, Nishimura N, Yao ZH. A fast multipole accelerated method of fundamental solutions for potential problems. Engineering Analysis with Boundary Elements 2005;29(11):1016–24.
[11] Young DL, Chen KH, Lee CW. Novel meshless method for solving the potential problems with arbitrary domain. Journal of Computational Physics 2005;209: 290–321.
[12] Chen KH, Kao JH, Chen JT, Young DL, Lu MC. Regularized meshless method for multiply-connected-domain Laplace problems. Engineering Analysis with Boundary Elements 2006;30:882–96.
[13] Young DL, Chen KH, Chen JT, Kao JH. A modified method of fundamental solutions with source on the boundary for solving Laplace equations with circular and arbitrary domains. CMES: Computer Modeling in Engineering & Sciences 2007;19(3):197–221.
[14] Šarler B. Solution of potential flow problems by the modified method of fundamental solutions: formulations with the single layer and the double layer fundamental solutions. Engineering Analysis with Boundary Elements 2009;33(12):1374–82.
[15] Liu YJ. Fast multipole boundary element method—theory and applications in engineering. Cambridge: Cambridge University Press; 2009.
[16] Chen W, Wang FZ. A method of fundamental solutions without fictitious boundary. Engineering Analysis with Boundary Elements 2010;34(5):530–2.
[17] Mukherjee S. Boundary element methods in creep and fracture. New York: Applied Science Publishers; 1982.
[18] Liu YJ, Rudolphi TJ. Some identities for fundamental solutions and their applications to weakly-singular boundary element formulations. Engineering Analysis with Boundary Elements 1991;8(6):301–11.