Smart Mater. Struct. **11** (2002) 308–311 PII: S0964-1726(02)34317-9

TECHNICAL NOTE

Boundary formulation and numerical analysis of elastic bodies with surface-bonded piezoelectric films

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Received 17 August 2001, in final form 4 February 2002
Published 5 April 2002
Online at stacks.iop.org/SMS/11/308

Abstract
The behavior of an elastic body with surface-bonded piezoelectric films is studied in this paper. The elastic body is modeled by the three-dimensional equations of elasticity, while the thin piezoelectric film is modeled by the two-dimensional equations of piezoelectric shells. It is shown that the governing equations can be written as a system of boundary integral–differential equations. These equations are solved numerically by the boundary element method in an example.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The mechanics of an elastic body coated with an elastic film of a different material was first studied in [1]. It is useful in surface and interface phenomena. Further development on the subject can be found in [2]. During the development of smart structures, thin elastic bodies coated with piezoelectric films have been studied extensively and many structural theories have been developed [3–6]. When the elastic bodes are not very thin, structural theories are not accurate enough and three-dimensional modeling of the elastic bodies is needed. In applications, three-dimensional (non-thin) bodies are widely used. The sensing of the deformation of three-dimensional bodies is important in, for example, nondestructive testing. New technologies also allow piezoelectric paint to be deposited on bodies of any shapes [7]. Recently, attempts have been made to attach artificial muscles [8] made from thin electroelastic films to human hearts, which are three-dimensional in nature. All these require the knowledge of the behaviour of a three-dimensional body coated with thin electroelastic films, either for sensing or for actuating. In this paper, the mechanics of a three-dimensional elastic body coated with a thin piezoelectric film is studied. The thin piezoelectric film is modeled by the two-dimensional equations of piezoelectric shells. It is shown that the behavior of the body and the film is governed by a set of boundary integral–differential equations. These equations are solved by the boundary element method (BEM) numerically in an example.

2. Equations for a thin piezoelectric shell

We consider a thin piezoelectric shell with the following material matrices under the compact matrix notation [9]:

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \quad \begin{pmatrix} e_{11} & e_{21} & e_{31} \\ e_{12} & e_{22} & e_{32} \\ e_{13} & e_{23} & e_{33} \\ e_{14} & e_{24} & e_{34} \\ e_{15} & e_{25} & e_{35} \\ e_{16} & e_{26} & e_{36} \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix}. \qquad (1)$$

As special cases, the matrices in (1) include commonly used piezoelectric materials such as ceramics poled in the 1, 2, or 3 directions which are transversely isotropic (∞mm), PVDF ($2mm$) and lithium tetraborate ($4mm$). The shell is assumed to be very thin so that the membrane theory for tension can be employed, which describes a film that does not resist bending or transverse shear. The three-dimensional mechanical displacements u_i and electric potential ϕ are related to the middle-plane displacements and potentials by

$$u_i = u_i(\alpha_1, \alpha_2), \qquad \phi = \phi^{(0)}(\alpha_1, \alpha_2) + \alpha_3 \phi^{(1)}(\alpha_1, \alpha_2), \tag{2}$$

where α_1 and α_2 are the middle-plane principal coordinates, α_3 is the thickness coordinate and $\phi^{(0)}$ and $\phi^{(1)}$ are the zeroth- and first-order electric potentials. The corresponding membrane strains and electric fields are given by [10]

$$\begin{aligned} S_1 &= \frac{1}{A_1}\frac{\partial u_1}{\partial \alpha_1} + \frac{u_2}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + \kappa_1 u_3, \\ S_2 &= \frac{1}{A_2}\frac{\partial u_2}{\partial \alpha_2} + \frac{u_1}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + \kappa_2 u_3, \\ S_6 &= \frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right) + \frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right), \\ E_1^{(0)} &= -\frac{1}{A_1}\frac{\partial}{\partial \alpha_1}(\phi^{(0)}), \qquad E_2^{(0)} = -\frac{1}{A_2}\frac{\partial}{\partial \alpha_2}(\phi^{(0)}), \\ E_3^{(0)} &= -\phi^{(1)}, \end{aligned} \tag{3}$$

where A_1 and A_2 are Lame's coefficients, and κ_1 and κ_2 are the principal curvatures of the middle surface of the shell. The membrane tensile and shear forces and electric displacements are given by the following membrane constitutive relations:

$$\begin{aligned} N_1 &= hc_{11}^p S_1 + hc_{12}^p S_2 - he_{k1}^p E_k^{(0)}, \\ N_2 &= hc_{12}^p S_1 + hc_{22}^p S_2 - he_{k2}^p E_k^{(0)}, \\ N_6 &= hc_{66} S_6 - he_{k6} E_k^{(0)}, \\ D_k^{(0)} &= he_{k1}^p S_1 + he_{k2}^p S_2 + he_{k6} S_6 - h\varepsilon_{kl}^p E_l^{(0)}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} c_{11}^p &= c_{11} - c_{13}^2/c_{33}, \qquad c_{12}^p = c_{12} - c_{13}c_{32}/c_{33}, \\ c_{22}^p &= c_{22} - c_{23}^2/c_{33}, \\ e_{k1}^p &= e_{k1} - c_{13}e_{k3}/c_{33}, \qquad e_{k2}^p = e_{k2} - c_{23}e_{k3}/c_{33}, \\ \varepsilon_{kl}^p &= \varepsilon_{kl} - e_{k3}e_{l3}/c_{33}. \end{aligned} \tag{5}$$

The membrane equations of equilibrium and electrostatics take the form [10]

$$\frac{\partial}{\partial \alpha_1}(A_2 N_1) + \frac{\partial}{\partial \alpha_2}(A_1 N_6) + N_6 \frac{\partial A_1}{\partial \alpha_2} - N_2 \frac{\partial A_2}{\partial \alpha_1} + A_1 A_2\big(\rho h f_1 + T_{31}|_{-h/2}^{h/2}\big) = 0, \tag{6a}$$

$$\frac{\partial}{\partial \alpha_1}(A_2 N_6) + \frac{\partial}{\partial \alpha_2}(A_1 N_2) + N_6 \frac{\partial A_2}{\partial \alpha_1} - N_1 \frac{\partial A_1}{\partial \alpha_2} + A_1 A_2\big(\rho h f_2 + T_{32}|_{-h/2}^{h/2}\big) = 0, \tag{6b}$$

$$-A_1 A_2 \kappa_1 N_1 - A_1 A_2 \kappa_2 N_2 + A_1 A_2\big(\rho h f_3 + T_{33}|_{-h/2}^{h/2}\big) = 0, \tag{6c}$$

$$\frac{\partial}{\partial \alpha_1}(A_2 D_1^{(0)}) + \frac{\partial}{\partial \alpha_2}(A_1 D_2^{(0)}) + A_1 A_2 D_3|_{-h/2}^{h/2} = 0, \tag{6d}$$

where h is the shell thickness, f_i is the body force, T_{ij} is the three-dimensional stress tensor and D_i is the three-dimensional dielectric displacement. With successive substitutions from (3) and (4), equations (6*a*)–(6*c*) can be written in the following compact form:

$$L_i(\boldsymbol{u}, \phi^{(0)}, \phi^{(1)}) + \rho h f_i + t_i^+ - t_i^- = 0, \tag{7}$$

where L_i are linear differential operators, and $t_i^\pm = T_{3i}(\pm h/2)$ are the traction vectors at the major faces of the film. At the edge of the film two mechanical boundary conditions of the membrane forces N_1, N_2 and N_6 and the membrane displacements u_1 and u_2 or their combinations need to be specified. For electrical boundary conditions one condition of $\phi^{(0)}$ or $D_1^{(0)}$ and $D_2^{(0)}$ or their combination should be specified.

3. Boundary formulation of the problem

We consider a three-dimensional elastic body in equilibrium, which is governed by the equations of linear elasticity. On the surface of the body a thin piezoelectric film is coated, which is governed by the equations in the previous section. Effectively these equations of the piezoelectric film appear in the boundary conditions of the differential equations of the elastic body. Since the film interacts with the body through the surface of the body, a more interesting and effective approach is through the boundary integral equation (BIE) formulation and its numerical solution technique—the BEM. The displacement $\boldsymbol{u}$ and surface traction $\boldsymbol{t}$ of the elastic body (in a domain Ω with boundary Γ) satisfy the following BIE [11]:

$$\boldsymbol{C}(P_o)\boldsymbol{u}(P_o) + \int_\Gamma \hat{\boldsymbol{T}}(P, P_o)\boldsymbol{u}(P)\,\mathrm{d}\Gamma(P) = \int_\Gamma \hat{\boldsymbol{U}}(P, P_o)\boldsymbol{t}(P)\,\mathrm{d}\Gamma(P) + \int_\Omega \hat{\boldsymbol{U}}(P, P_o)\boldsymbol{b}(P)\,\mathrm{d}\Omega(P), \tag{8}$$

where $C_{ij} = \delta_{ij}/2$ for a smooth boundary, $\boldsymbol{b}$ is the body force vector, $\hat{\boldsymbol{U}}$ and $\hat{\boldsymbol{T}}$ are known second-rank tensors and are related to the fundamental solution of the Navier operator, P_o is the source point and P is the field point. The traction vector $\boldsymbol{t}$ on the surface of the elastic body is related to the traction on one of the faces of the film by $\boldsymbol{t} = -\boldsymbol{t}^-$. Therefore, from (7) and (8), we obtain

$$\boldsymbol{C}\boldsymbol{u} + \int_\Gamma \hat{\boldsymbol{T}}\boldsymbol{u}\,\mathrm{d}\Gamma = -\int_\Gamma \hat{\boldsymbol{U}}(\boldsymbol{L} + \rho h \boldsymbol{f} + \boldsymbol{t}^+)\,\mathrm{d}\Gamma + \int_\Omega \hat{\boldsymbol{U}}\boldsymbol{b}\,\mathrm{d}\Omega, \tag{9}$$

where the displacement continuity conditions between the body and the film have been used. Equation (9) is a system of boundary integral–differential equations because of the differential operator $\boldsymbol{L}$. If the body is only partially coated with a film, then in the non-coated portion of the surface BIE (8) still applies. Boundary integral equations [12, 13] and boundary integral–differential equations [14] have been used to study piezoelectric films bonded to the surface of two-dimensional semi-infinite elastic bodies. Equation (9) represents a generalization of the equations in [12–14] to three-dimensional finite bodies. Equation (9) is in fact rather complicated. In the following we examine the basic behavior of an elastic body with a piezoelectric film governed by (9) in an example.

4. A two-dimensional example

Consider a two-dimensional plane-strain example of a circular elastic body of radius R shown in figure 1. From $-\alpha$ to α, the body is coated with a ceramic film of thickness h which is poled in the thickness direction. The film is electroded at its two major faces and can work as either a sensor or an actuator. The voltage across the electrodes is denoted by V, which implies, from (2),

$$\phi^{(0)} = 0, \qquad \phi^{(1)} = V/h. \tag{10}$$

The relevant membrane force, in the polar-coordinate system, is

$$N_\theta = hc_{11}^p(u_{,\theta} + u_r)/R + e_{31}^p V, \tag{11}$$

where, for ceramics, we have

$$c_{11}^p = c_{11} - c_{13}^2/c_{33}, \qquad e_{31}^p = e_{31} - c_{13}e_{33}/c_{33}. \tag{12}$$

The traction on the elastic body can be determined from (6*a*)–(6*d*) as

$$t_r = -\frac{1}{R}N_\theta = -\frac{1}{R}\left(hc_{11}^p\frac{u_{,\theta}+u_r}{R} + e_{31}^p V\right) \cong -\frac{1}{R}e_{31}^p V,$$
$$t_\theta = \frac{1}{R}\frac{\partial N_\theta}{\partial\theta} = \frac{1}{R}\left(hc_{11}^p\frac{u_{,\theta\theta}+u_{r,\theta}}{R} + e_{31}^p V_{,\theta}\right) \cong \frac{1}{R}e_{31}^p V_{,\theta}, \tag{13}$$

where the approximation is for the case when the film is relatively soft compared with the body (small c_{11}^p). Since V is a piecewise constant function, its derivative leads to a delta function and the traction is effectively a normal distribution q and a pair of concentrated forces Q as shown in figure 2 with

$$q = e_{31}^p V/R, \qquad Q = e_{31}^p V. \tag{14}$$

The presence of concentrated Q can also be seen from the boundary condition of vanishing N_θ at the edge of the film and (11). We note that q is related to the curvature of the shell and does not exist for a plate, while Q is the same as the actuating force of a plate actuator [15]. When the film is not soft compared with the body, Q represents the resultant of a narrow local distribution [15]. If the global response rather than the local stress distribution is of main interest, (14) can still be used. Then effectively the traction on the elastic body due to the film is known and the usual boundary element analysis of an elastic body can be performed. Otherwise boundary integral–differential equations will need to be solved, which can be done numerically as in the special case of a two-dimensional half-space in [14].

As a numerical example we consider a PZT-7A piezoelectric film with the following material constants [16]:

$$\rho = 7500\ \mathrm{kg\ m^{-3}}, \quad c_{11} = c_{22} = 148, \quad c_{33} = 131,$$
$$c_{12} = 76.2, \quad c_{13} = c_{23} = 74.2, \quad c_{44} = c_{55} = 25.4,$$
$$c_{66} = 35.9\ \mathrm{GPa}, \quad e_{15} = 9.2, \quad e_{31} = -2.1,$$
$$e_{33} = 9.5\ \mathrm{C\ m^{-2}}, \quad \varepsilon_{11} = 460\varepsilon_0, \quad \varepsilon_{33} = 235\varepsilon_0,$$
$$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{F\ m^{-1}}. \tag{15}$$

For geometric parameters we choose $R = 20$ mm and $h = 1$ mm. As an example of an actuator a voltage $V = 10$ V is applied across the film. For the elastic body we consider plastics with $E = 2.0 \times 10^9$ Pa and $\nu = 0.3$. The BIE/BEM code developed in [17] is employed for this study and 72 quadratic boundary elements are used (figure 3). The three nodes at the (0, 20), (−20, 0) and (0, −20) locations in the BEM model are fixed in the tangential direction. The deformed shapes of the body under the voltage are shown in figures 4 and 5 for two different values of α. The deformed shapes are as expected under the applied loads and constraints.

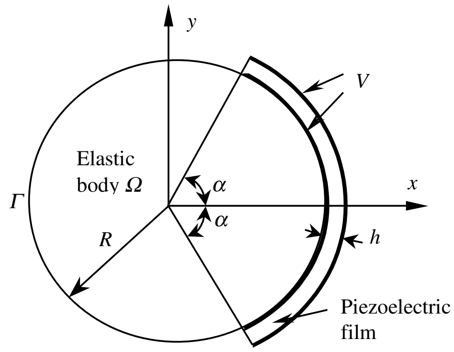

Figure 1. An elastic body coated with a piezoelectric film. Thick curves represent electrodes.

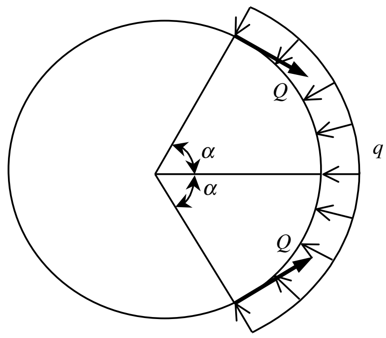

Figure 2. Actuating forces on the elastic body due to the film.

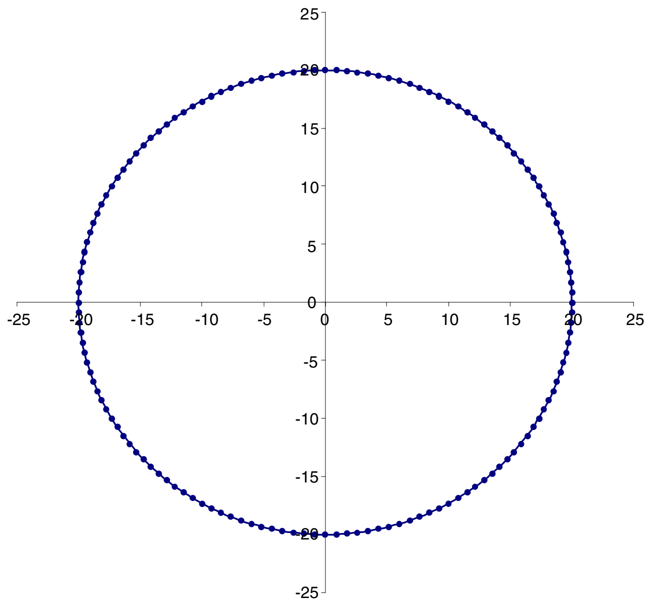

Figure 3. Discretization of the circular elastic body with 72 quadratic boundary elements (three nodes form one element).

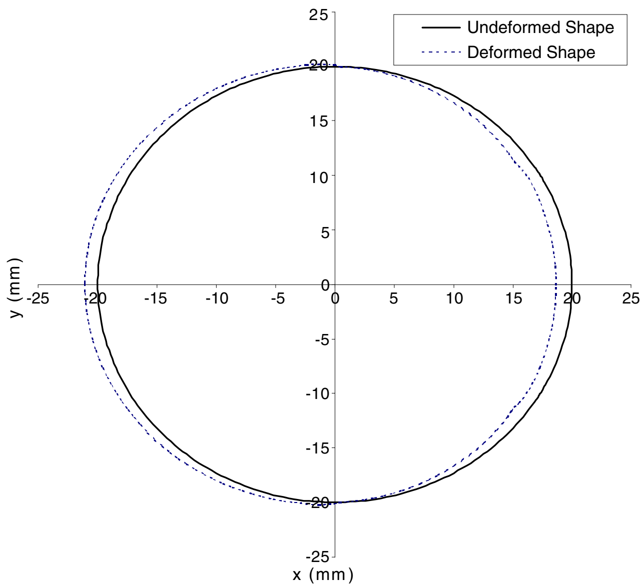

Figure 4. Deformed shape of the elastic body ($\alpha = 30°$).

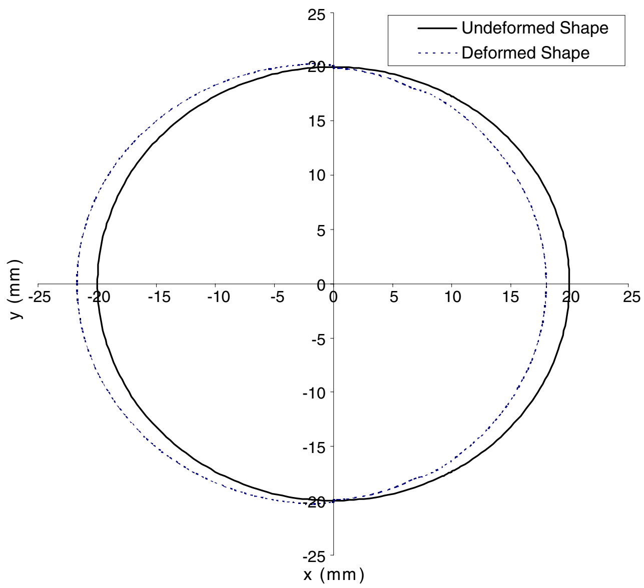

Figure 5. Deformed shape of the elastic body ($\alpha = 60°$).

Conversely, if the body is deformed due to mechanical loads, a voltage will be produced across the electrodes of the film as a sensing signal. The above results are based on the approximation in (13). Direct numerical computation based on (9) is left as future work. The present boundary formulation can be generalized to include the bending effect of the piezoelectric film, as in the elastic case [2].

It should be pointed out that the BEM is very efficient in solving problems where the boundary solutions (displacement and stresses) are of paramount interest. The BEM discretizes the boundary and interfaces of the problem domain only and provides the solutions where they are most desired. This is in strong contrast to domain-based methods, such as the finite-element method, that discretize the whole domain and provide the solutions for all locations.

5. Conclusion

The behavior of an elastic body coated with thin piezoelectric films can be described by a boundary formulation. The boundary element technique can be an efficient method to solve the problem numerically. The formulation and solution technique are useful in the sensing and actuating of elastic bodies through surface-bonded piezoelectric films. The present formulation can be generalized to include more sophisticated behavior of the piezoelectric films using higher-order theories of shells.

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