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On the simple-solution method and non-singular nature of the BIE/BEM — a review and some new results

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Abstract

The non-singular nature of the boundary integral equations (BIEs) in the boundary element method (BEM) is discussed in this paper. The research effort leading to this conclusion is reviewed, tracing back to the early, classical BIE/BEM works in the 1960s and 1970s. It is believed that Cruse's 1974 paper published in Computers & Structures was the first work which demonstrates how the free term coefficient (C_{ij}) for the elastostatic BIE can be expressed by an integral using the rigid-body motion (a simple solution). This concept later led to the simple-solution method, and identities for fundamental solutions, developed to regularize various BIEs, including the hypersingular ones. New results in the identities for fundamental solutions are presented and their applications in the BIE formulations are illustrated. Recent work to show that the BIEs can be further regularized to completely non-singular forms is also discussed.

Keywords: Simple-solution method; Integral identities; Non-singular BIEs

1. Singular or non-singular? The early approach

The boundary integral equation/boundary element method (BIE/BEM), pioneered by Jaswon and his colleagues for potential problems [1–3], by Rizzo for elastostatic problems [4] and by Cruse et al. for elastodynamic, 3D stress, fracture and elasto-plastic problems [5–10], has experienced almost four decades of development. It is now applicable to almost all areas of engineering as a powerful, alternative numerical tool. However, singularities in the BIEs, arising from the applications of the singular fundamental solutions in BIE formulations, are still regarded by many as drawbacks in the BIE/BEM or at least difficult to cope with. A great deal of research efforts have been devoted to the subject of how to compute accurately and efficiently the various singular integrals in the BIE/BEM either numerically or analytically in the last four decades. Although these efforts, which are continuing even today, have led to the development of many useful integration techniques for singular integrals, they turn out to be not a necessity for the development of the BEM. In fact, as will be discussed in this paper, the direct computation of various singular integrals can be avoided altogether in the BIE/BEM in most cases, if the weakly singular or non-singular forms of the BIEs are employed, with greater efficiency and without sacrificing any accuracy. This can be achieved in the BEM because the BIEs for most problems *do not* contain singular integrals at all if they are formulated properly, even if the fundamental solutions employed in BIEs are in general singular.

To trace back the historical development of the non-singular BIEs, let us first consider the direct BIE for elastostatic problems [4] as given below (index notation is applied in this paper):

$$C_{ij}(P_0)u_j(P_0) = \int_S [U_{ij}(P,P_0)t_j(P) - T_{ij}(P,P_0)u_j(P)]\mathrm{d}S(P), \quad \forall P_0 \in S, \tag{1}$$

in which u_i and t_i are the displacement and traction fields, respectively; u_{ij} and T_{ij} the displacement and traction components, respectively, of the Kelvin (fundamental) solution in elasticity; P and P_0 the field and source points, respectively; V the domain of the body and S the boundary (Fig. 1). The kernel T_{ij} is singular at the source point P_0 with the order of singularity $O(1/r^2)$ in 3D or $O(1/r)$ in 2D (r is the distance between P_0 and P, Fig. 1); while the kernel U_{ij} is weakly singular with the order $O(1/r)$ in 3D or $O(\ln(1/r))$ in 2D.

BIE (1) is definitely singular (or in a singular form) if the explicit values of the free term coefficient C_{ij} (containing a solid-angle integral) are employed. For example, one can use $C_{ij} = (1/2)\delta_{ij}$ (δ_{ij} is the Kronecker symbol) in BIE (1) for source point P_0 on boundary S where it is smooth. This has been the practice in most BIE work and thus initiated the

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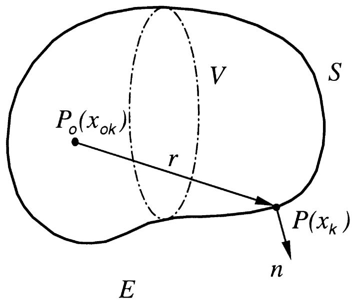

Fig. 1. A closed domain V in R^3 with boundary S (exterior domain $E = R^3 - (V \cup S)$).

need of research on how to evaluate accurately and efficiently the singular integrals containing the T_{ij} kernel in BIE (1) for either 2- or 3D problems with various types of boundary element. In the early days of the BIE/BEM development [1–10], this form of the BIE was no hindrance to the applications of the BEM, since the problems considered were 2D and/or constant boundary elements were employed, where boundary segments are always smooth at the collocation point P_0 (center of the element). Under these conditions, expression $C_{ij} = (1/2)\delta_{ij}$ is sufficient and singular integrals can also be evaluated analytically. However, as the numerical procedures for the BIEs became more sophisticated, the singular form of BIE (1) with an expression such as $C_{ij} = (1/2)\delta_{ij}$ was no longer conveniently available, for example, for 3D geometries with corners and edges, or employing linear [11] or quadratic elements [12,13]. A more general approach to express or evaluate the free term C_{ij} would be desirable.

Cruse [11] was the first to introduce the alternative (indirect) way to evaluate the free term C_{ij}. He applied a translational rigid-body motion (a simple solution) to BIE (1) and obtained an expression for this free term [11] that can now be written as:

$$C_{ij}(P_0) = -\int_S T_{ij}(P, P_0)\mathrm{d}S(P), \qquad \forall P_0 \in S, \tag{2}$$

which is itself a singular integral in the sense of Cauchy-principal value (CPV). Cruse pointed out in Ref. [11] that this integral for C_{ij} can be obtained by summing the already computed integrals containing the same T_{ij} kernel on all other elements. Thus, the use of expression (2) for the free term does not endure any penalty in the BEM procedure. This approach in evaluating the free term is quite general, applicable to all geometries (with corners and edges) and all types of boundary element employed. Direct evaluation of the C_{ij} coefficients, although attempted later by many researchers for various geometric settings, is actually not needed for the BEM process. In fact, this author believes that the method for computing the diagonal submatrices by summing the off-diagonal submatrices for the final BEM matrix associated with the singular kernel, as presented in later BEM textbooks (see, for example, Refs. [14,15]), can be traced back to this earlier observation by Cruse on computing the C_{ij} term using expression (2) and the summation of off-diagonal coefficients.

There is another important implication of expression (2) for the free term C_{ij}, which was unfortunately not explicitly expressed in Ref. [11]. That is, if one substitutes expression (2) back into BIE (1) and rearranges the terms, one can arrive at the following explicit *weakly singular form* of the BIE for elastostatic problems:

$$\int_S T_{ij}(P, P_0)[u_j(P) - u_j(P_0)]\mathrm{d}S(P) = \int_S U_{ij}(P, P_0)t_j(P)\mathrm{d}S(P),$$
$$\forall P_0 \in S, \tag{3}$$

in which the integral on the left-hand side is only weakly singular, due to the subtraction using the one-term Taylor's series expansion of the density function (displacement). Eq. (3) is valid for a *finite* domain. What happened in the rearrangement of the terms is that the two CPV integrals, one in the C_{ij} expression (2) and one in the original form of the BIE as shown in Eq. (1), are cancelled out naturally and completely. Therefore, there is no strongly singular integrals in BIE (3) anymore, which can significantly reduce the burden on the BEM implementations, as has been demonstrated in the literature. A polar-coordinate transformation on a surface element will further remove the remaining weak singularities in the two integrals in BIE (3) for 3D problems and the regular Gaussian quadrature can be applied thereafter [13].

Rizzo and Shippy [13] were the first to write the elastostatic BIE (including thermal loading) in a form equivalent to Eq. (3) by employing the one-term subtraction of the displacement field and expression (2) to evaluate the free term. There are no singular integrals ever computed, either numerically or analytically, in the work reported in [13]. This weakly singular-integral-only approach was later extended to acoustics [16] and elastodynamics [17], and has been the philosophy in the BIE/BEM work of many research groups.

The process leading to the weakly singular form (Eq. (3)) of the elastostatic BIE demonstrates that the two strongly singular integrals in the BIE can be cancelled out completely, if they have been identified. The utilization of the fundamental solutions, which are the origin of the singularity, in the BIEs does not necessarily give rise to singular BIE formulations for the physical problems which, in most cases, are not singular at all in the first place.

2. Generalization of the simple-solution method

The weakly singular nature of the BIEs is quite general, not just limited to the conventional BIE formulations of elastostatic, acoustic or elastodynamic problems. In Ref. [18], Rudolphi systematically generalized the simple-solution approach and obtained two integral identities for the fundamental solutions in potential problems. He successfully applied these two identities, for the first time, in the derivations of the weakly singular form of the hypersingular BIE (derivative of the conventional BIE) for potential problems. In this simple-solution approach, simple solutions (or modes of the problem, such as the rigid-body translation and rotation in the elastostatic case) are imposed on the BIEs and result in certain identities for the fundamental solutions, which are then applied to regularize the singular or hypersingular BIEs. This simple-solution concept can be readily extended to elasticity and even nonlinear problems.

A more general way to establish the identities for the fundamental solutions was developed by Liu and Rudolphi in Ref. [19] based on a so-called *operational approach*. In this approach the governing equations for the fundamental solutions are integrated over an arbitrary closed domain, and Gauss' theorem and the properties of the Dirac-delta function (representing the unit source) are employed to transform the domain integrals to boundary ones. This approach, although involving the concept of generalized functions, does not depend on the availability of the BIE formulations of the problems, offers more physical insights to the identities for the fundamental solutions (equilibrium of the forces, moments, etc.) and is applicable to both *finite* and *infinite* domains. The weakly singular forms of the conventional and hypersingular BIEs for potential as well as elastostatic problems, in both finite and infinite (2- and 3D) domains, were established readily in Ref. [19] using these identities.

It was also shown in Ref. [19], perhaps for the first time, that the discretization of the weakly singular form of the BIE (e.g. Eq. (3)), leads directly to the results that the diagonal submatrices in the matrix associated with the strongly singular kernel T_{ij} are determined by the summation of the off-diagonal submatrices. That is,

$$\mathbf{T}_{ii} = \begin{cases} -\sum_{j\neq i} \mathbf{T}_{ij}, & \text{for a finite domain;} \\ \mathbf{I} - \sum_{j\neq i} \mathbf{T}_{ij}, & \text{for an infinite domain,} \end{cases} \qquad (4)$$

where $\mathbf{T}_{ij}$ is the submatrix and $\mathbf{I}$ the identity matrix. Thus, the method of summing the off-diagonal submatrices to obtain the diagonal ones, initiated in Ref. [11] and presented in a more straightforward way (imposing the simple solution directly to the discretized BEM equations) in Refs. [14,15], is fully consistent with the BIE formulation. There is no approximation involved in this process and the practice of calculating the singular integrals directly using either analytical or numerical methods does not provide any additional benefit in accuracy, as has been pointed out in Ref. [19].

Use of the identities for fundamental solutions in deriving the weakly singular forms of the BIEs, especially those of the hypersingular BIEs [20–24], has been employed successfully in potential problems [18,19,25,26], elastostatics [19,27–31], acoustics [32,33], elastodynamics [34,35], electromagnetics [36] and many other problems (see, e.g. the review papers in Refs. [37,38]). Recently, this simple solution approach was successfully extended to nonlinear problems in Ref. [39] for regularizing the hypersingular BIE in elastoplasticity, and in Ref. [40] for regularizing the hypersingular BIE for thermoelastic fracture mechanics problems.

Recently, Liu and Rudolphi [41] further showed that the BIEs for potential and elastostatic problems can be written in non-singular forms, which do not contain even the weakly singular integrals, by employing additional integral identities for the fundamental solutions. For example, the conventional BIE (1) for elastostatic problems can be written in the following *non-singular* form [41]:

$$\int_S T_{ij}(P,P_0)[u_j(P) - u_j(P_0) - u_{j,k}(P_0)(x_k - x_{0k})]\mathrm{d}S(P)$$
$$= \int_S U_{ij}(P,P_0)[\sigma_{jk}(P) - \sigma_{jk}(P_0)]n_k(P)\mathrm{d}S(P),$$
$$\forall P_0 \in S, \qquad (5)$$

for a *finite* domain V (Fig. 1), where σ_{ij} is the stress field. A free term $u_i(P_0)$ needs to be added to the left-hand side of Eq. (5) if it is applied to an *infinite* domain [41]. The weak and strong singularities in the two kernels U_{ij} and T_{ij}, respectively, have been cancelled out due to the use of Taylor's series expansions for the two density functions. Although this non-singular form of the BIE may not have the merit suitable for the BEM procedure, its existence however further enhances the argument that the singularities in the BIEs are nonessential and removable [19,42].

From the initial strongly singular form (Eq. (1)) of the conventional BIE for elastostatic problems, to the weakly singular form (Eq. (3)) and recently to the non-singular form (Eq. (5)), the simple-solution concept or applying identities (properties) satisfied by the fundamental solutions, initiated in Ref. [11] and explored fully in Refs. [18,19,41], played a crucial role in revealing the non-singular nature of the BIEs.

3. Review of the identities for fundamental solutions

The four integral identities developed so far for the fundamental solution in elastostatics are reviewed in this section. Applications of these identities in regularizing the various BIE formulations and their advantages are discussed. New results (extensions) of these identities and the implications will be presented in Section 4.

Consider an arbitrary, closed domain V in the infinite space R^m, with $m = 2$ or 3 for 2- or 3D space, respectively (Fig. 1). The following four integral identities for the fundamental solution $U_{ij}(P,P_0)$ (Kelvin solution) in elastostatics can be established:

The first identity [11,13,19]:

$$\int_S T_{ij}(P,P_0)\mathrm{d}S(P) = \begin{cases} -\delta_{ij}, & \forall P_0 \in V, \\ 0, & \forall P_0 \in E; \end{cases} \tag{6}$$

the second identity [19]:

$$\int_S \frac{\partial T_{ij}(P,P_0)}{\partial x_{0k}}\mathrm{d}S(P) = 0, \qquad \forall P_0 \in V \cup E; \tag{7}$$

the third identity [19]:

$$E_{jlpq}\int_S \frac{\partial U_{iq}(P,P_0)}{\partial x_{0k}} n_p(P)\mathrm{d}S(P) - \int_S \frac{\partial T_{ij}(P,P_0)}{\partial x_{0k}}(x_l - x_{0l})\mathrm{d}S(P) = \begin{cases} \delta_{ij}\delta_{kl}, & \forall P_0 \in V, \\ 0, & \forall P_0 \in E; \end{cases} \tag{8}$$

and the fourth identity [41]:

$$\int_S T_{ij}(P,P_0)(x_k - x_{0k})\mathrm{d}S(P) - E_{jkpq}\int_S U_{ip}(P,P_0)n_q(P)\mathrm{d}S(P) = 0, \qquad \forall P_0 \in V \cup E; \tag{9}$$

where E_{ijkl} is the elastic modulus tensor and δ_{ij} the Kronecker delta. These four identities for the elastostatic fundamental solution can be derived by using the *operational approach* [19,41], that is, by integrating the following governing equations (or their derivatives) for the fundamental solution and applying the properties of Dirac δ-function $\delta(P,P_0)$ representing the unit source:

$$\Sigma_{ijk,k}(P,P_0) + \delta_{ij}\delta(P,P_0) = 0, \qquad \forall P, P_0 \in R^m, \tag{10}$$

where $\Sigma_{ijk}(P,P_0) = E_{jkpq}U_{ip,q}(P,P_0)$ is the stress tensor in the fundamental solution. In general, all the integral identities for the fundamental solution in elastostatics can be derived by starting with the following integration of governing Eq. (10) over the domain V:

$$\int_V (x_p - x_{0p})^\alpha \frac{\partial^\beta}{\partial x_{0q}^\beta}\left[\Sigma_{ijk,k}(P,P_0) + \delta_{ij}\delta(P,P_0)\right]\mathrm{d}V(P) = 0, \qquad \forall P_0 \in V \cup E, \tag{11}$$

where $\alpha, \beta = 0, 1, 2, 3, \ldots$. The first four identities derived so far (Eqs. (6)–(9)) are corresponding to the combinations of $\alpha, \beta = 0$ or 1. A similar starting integral expression exists for potential problems [41]. Four similar identities for the fundamental solution in potential problems are provided in Refs. [19,41].

The integral identities in Eqs. (6)–(9) for the elastostatic problem, in the cases when $P_0 \in V$, can also be derived by imposing certain types of simple solution, such as rigid-body translations, to the corresponding representation integrals [11,13,19,41]. However, the operational approach developed in Refs. [19,41] and based on the original governing equations seems to be more general and offers more insights into the BIE formulations. Indeed, the operational approach does not depend on the existence of the integral equation formulations and the explicit expressions of the fundamental solutions. The physical meaning (equilibrium of forces, moments, and so on [19]) of these integral identities can also be recognized readily from their derivations using this operational approach. Finally, the operational approach is applicable to both *finite* and *infinite* domain problems, while the simple-solution approach is limited only to a finite domain problem since the rigid-body motion cannot be imposed to an infinite domain directly.

The first identity (6) can be used to regularize the conventional BIE (1) to weakly singular form (3) [11,13,19]; the second and third identities (7) and (8) can be employed to regularize the hypersingular (traction) BIE [19,27,34], and identity (9) can be applied to regularize BIE (1) to the non-singular form (5) [41]. Using these identities offers a general and systematic approach to the development of the weakly singular or even non-singular forms of the BIEs, as compared to the earlier approach where the explicit expressions of the fundamental solutions need to be exploited in great length in order to cancel the singularities in the BIEs (see, e.g. Refs. [28,43]).

4. Extension of the identities for fundamental solutions for $P_0 \in S$

In the four identities for fundamental solutions, as presented in Eqs. (6)–(9) for elastostatics, the source point P_0 can only be either within the enclosed boundary $S(P_0 \in V)$ or outside the boundary $S(P_0 \in E)$, see Fig. 1. Although these results are sufficient for the purposes of regularizing the singular or hypersingular BIEs, one is always curious about these identities when the source point is right on the boundary $S(P_0 \in S)$ and the possible applications of such results in the BIE formulations.

In fact, the first result derived by Cruse [11], i.e. Eq. (2), for the free-term coefficient $C_{ij}(P_0)$, is the result of the first identity (6) with the source point on the boundary S. However, this result (Eq. (2)) was derived by imposing the simple solution to the BIE (Eq. (1)). Therefore, this approach depends on the availability of the BIEs. In the following, we will derive this same identity (Eq. (2)) for source point on the boundary S by using the *operational approach* and without using BIE (1). In this way, the result will be shown to be independent of the existence of BIE (1).

Extension of the identities is possible to include the case $P_0 \in S$, by using the operational approach, at least for the first identity (6) and when S is smooth at the source point P_0. To demonstrate this, the following sifting property of the Dirac δ-function [44] in 2 or 3D needs to be established:

$$\int_S f(P)\delta(P,P_0)\mathrm{d}S(P) = \begin{cases} f(P_0), & \forall P_0 \in V, \\ \frac{1}{2}f(P_0), & \forall P_0 \in S(\text{smooth}), \\ 0, & \forall P_0 \in E, \end{cases} \tag{12}$$

where $f(P)$ is an arbitrary and continuous function. Note that the integral for $P_0 \in S$ is a CPV integral and its result $(\frac{1}{2}f(P_0))$ can be verified by considering the following relations, for example, in 1D [44]:

$$\delta(x,x_0) = \frac{\mathrm{d}}{\mathrm{d}x}H(x,x_0), \text{ and } H(x,x_0) = \begin{cases} 1, & x > x_0, \\ \frac{1}{2}, & x = x_0, \\ 0, & x < x_0; \end{cases} \tag{13}$$

where $H(x,x_0)$ is the Heaviside unit-step function [44] and x_0 a reference (source) point. Thus, one can derive, using the integration by part, for the function $f(x)$ on an interval (a,b):

$$\int_a^b f(x)\delta(x,x_0)\mathrm{d}x = \int_a^b f(x)\frac{\mathrm{d}H(x,x_0)}{\mathrm{d}x}\mathrm{d}x = f(x)H(x,x_0)\Big|_a^b - \int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x}H(x,x_0)\mathrm{d}x$$

$$= \begin{cases} f(b)H(b,a) - f(a)H(a,a) - \int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x}1\mathrm{d}x = f(b) - f(a)\frac{1}{2} - f(b) + f(a) = \frac{1}{2}f(a), & \text{if } x_0 = a; \\ f(b)H(b,b) - f(a)H(a,b) - \int_a^b \frac{\mathrm{d}f(x)}{\mathrm{d}x}0\mathrm{d}x = \frac{1}{2}f(b), & \text{if } x_0 = b. \end{cases}$$

Then, combining this result with the familiar results when x_0 is either inside or outside the interval, one can write:

$$\int_a^b f(x)\delta(x,x_0)\mathrm{d}x = \begin{cases} f(x_0), & \text{for } x_0 \in (a,b); \\ \frac{1}{2}f(x_0), & \text{for } x_0 = a \text{ or } b; \\ 0, & \text{for } x_0 \notin [a,b]; \end{cases} \tag{14}$$

in which $[a,b]$ indicates the closed interval. This is the result for the 1D case corresponding to Eq. (12) for the 2 and 3D cases.

Now, if one integrates the governing Eq. (10) over the domain V and applies the Gauss theorem and formula (12), one can obtain the following extended result for the first identity (6):

$$\int_S T_{ij}(P,P_0)\mathrm{d}S(P) = \begin{cases} -\delta_{ij}, & \forall P_0 \in V, \\ -\frac{1}{2}\delta_{ij}, & \forall P_0 \in S\,(\text{smooth}), \\ 0, & \forall P_0 \in E, \end{cases} \tag{15}$$

where $T_{ij} = \sum_{ijk} n_k$ has been applied. Again, the integral for $P_0 \in S$ must be interpreted as a CPV integral. As discussed in Ref. [19], the physical meaning of this identity is that the tractions on S are in equilibrium with the unit concentrated

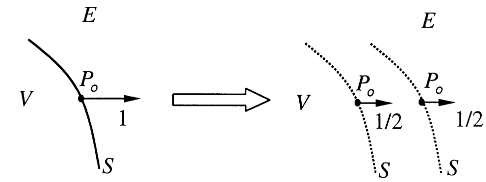

Fig. 2. The unit concentrated force split into two halves at the boundary S.

force applied at P_0. When this unit force is applied at a point P_0 on S, this unit force is split into two halves for a smooth S (see Fig. 2), thus the coefficient (1/2) in Eq. (15). If S is not smooth at the source point P_0, then a different value other than (1/2) should be used.

Extensions of the second, third and fourth identities (7)–(9) to include the results for $P_0 \in S$ are possible. However, the second and third identities (7) and (8) will involve the hypersingular integrals in this case, which may have to be interpreted in the Hadamard finite part sense. This is an interesting but challenging task. Here we limit our attention to results in Eq. (15) for the first identity for a smooth boundary S at source point P_0 and focus on the interesting consequences this extension brings to the BIE formulation.

5. Jump terms of the singular integral in BIE (1)

Evaluation of the *jump* terms of the singular integrals in the limit as the source point approaching the boundary is necessary in cases in which the BIE characteristics need to be studied, such as in the study of degeneracy or non-degeneracy of the BIEs for crack problems or thin shell-like structures [20,45,46].

With the extended results (Eq. (15)) for the first identity, established *independently* of BIE (1), one can derive the jump terms of the singular integral in BIE (1) for elastostatics using the classical limit-to-the-boundary approach, *without*, however, employing the explicit expressions of the fundamental solution. This is of non-trivial significance in the cases for which explicit expressions of the fundamental solutions do not exist, such as in the 3D BIE formulations for piezoelectric materials used in smart structures [47–49].

We can establish the results for the *jump* terms readily by applying the identity (15) for the fundamental solution, without exploiting the explicit expressions of the fundamental solution. The results for the jump terms are as follows:

$$\lim_{P_0 \to S} \int_S T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P)$$

$$= \int_s T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P) - \tfrac{1}{2} u_i(P_0),$$

$$\forall P_0 \in S \text{ (smooth)}, \tag{16}$$

when P_0 approaches S in the *same* direction of the normal n; and

$$\lim_{P_0 \to S} \int_S T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P)$$

$$= \int_s T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P) + \tfrac{1}{2} u_i(P_0),$$

$$\forall P_0 \in S \text{ (smooth)}, \tag{17}$$

when P_0 approaches S in the *opposite* direction of the normal n, where the first integral on the right-hand side of Eqs. (16) and (17) is a CPV integral. To prove Eq. (16), we consider the configuration shown in Fig. 1, that is, with P_0 approaching S from inside. In this case, we employ the classical approach for the limiting process [14,15] to obtain

$$\lim_{P_0 \to S} \int_S T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P)$$

$$= \lim_{\epsilon \to 0} \int_{S - S_\epsilon} T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P) + \lim_{\epsilon \to 0} \int_{S_\epsilon} T_{ij}(P, P_0) \mathrm{d}S(P) u_j(P_0)$$

$$= \int_S T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P) + \lim_{\epsilon \to 0} \left[\int_{(S - S_\epsilon) \cup S_\epsilon} T_{ij}(P, P_0) \mathrm{d}S(P) - \int_{S - S_\epsilon} T_{ij}(P, P_0) \mathrm{d}S(P) \right] u_j(P_0)$$

$$= \int_S T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P) + [-\delta_{ij} - (-\tfrac{1}{2}\delta_{ij})] u_j(P_0)$$

$$= \int_S T_{ij}(P, P_0) u_j(P) \mathrm{d}S(P) - \tfrac{1}{2} u_i(P_0),$$

$$\forall P_0 \in S \text{ (smooth)};$$

where S_ϵ is an outward hemisphere (bump) with radius ϵ and centered at P_0 placed on S, and the results in identity (15) have been applied. The resulting integral is a CPV integral. This proves Eq. (16). Result (17) can be proved similarly by using identity (15). Note that the surface S in Eqs. (16) and (17) can be a closed surface or an open surface. In the case of an open surface, an auxiliary surface can be introduced to form a closed one in order to apply the identity (15) which can be established for closed surfaces only.

Although the results (16) and (17) for the jump terms are not new, the way in which they are established as shown above is quite interesting. Here, the explicit expressions for the kernel $T_{ij}(P, P_0)$ is not used at all. The application of identity (15) with the source point on the boundary S can avoid the tedious task of evaluating the jump terms explicitly, which is only possible when explicit expressions for the kernels are available. When explicit expressions of the fundamental solutions are not available, such as for 3D piezoelectric solids [47–49], the above approach in evaluating the jump terms will be very valuable.

6. Conclusion

The simple-solution concept, applied first to the BIE for elastostatics by Cruse in Ref. [11] in 1974 to evaluate the free-term coefficient, is reviewed in this paper. The simple-solution method, or the identities for fundamental solutions, developed later are natural extensions of this simple, yet powerful concept. The approach to regularize the BIEs using these simple-solutions or identities clearly demonstrates that the strongly singular and hypersingular integrals in the various BIE formulations can be removed analytically. Thus, the BIEs can be recast in weakly singular and even non-singular forms. Singular integrals in the BIE should not be a problem in the applications of the BEM. Properties of the fundamental solutions, as represented by the identities, play an important role in achieving the regularization. Strongly singular and hypersingular integrals can be cancelled out naturally and completely, from both sides of the BIE formulations by exploiting these identities. Singularities in the fundamental solutions are deceiving and do not lead necessarily to singularities of the BIE formulations, if the properties of the fundamental solutions have been examined carefully and utilized, as was demonstrated elegantly in the earlier work by Cruse [11].

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