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TECHNICAL NOTE

Analysis of the shear stress transferred from a partially electroded piezoelectric actuator to an elastic substrate

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Abstract. The shear stress distribution between a partially electroded thin piezoelectric film and a semi-infinite elastic substrate is analyzed. It is shown that the shear stress is governed by a pair of integro-differential equations. These equations are solved numerically using the concept of elements and the finite difference method. A few special cases are treated analytically. The behavior of the shear stress is examined. The developed approach and results are useful in analyzing and understanding the behaviors of piezoelectric actuators on an elastic structure.

1. Introduction

The problem of a thin film on a substrate has been studied in quite a few ways for different applications. The film-edge-induced stress in a substrate due to intrinsic stresses in the film is studied in [1]. Thermal stress in the substrate due to a heated film is analyzed in [2], which is of fundamental importance in electronic packaging. The stress distribution in a piezoelectric half space by an applied electric potential on a surface electrode is obtained in [3], which has applications in the electronics industry and smart structures. The shear stress distribution between a piezoelectric film and an elastic substrate is calculated in [4]. The shear stress between a piezoelectric actuator and an elastic beam is given in [5]. This shear stress is the actuating stress of piezoelectric actuators.

In both [4] and [5], the piezoelectric film is fully electroded at its two major surfaces. A concentration of shear stress is predicted near the edge of the piezoelectric film. This shear stress concentration is one of the major causes of the undesired peeling-off of actuators from host structures. General equations for thin piezoelectric plates valid for electroded and unelectroded plates have been recently obtained in [6]. With these equations it is shown in [7], from plate theory, that if small portions near the edges of piezoelectric actuators are left unelectroded (partially electroded actuators), the shear stress concentration can be greatly reduced while the actuating effectiveness of the actuators remains the same. The same conclusions have also been reached in a more detailed analysis from the three-dimensional theory [8]. Therefore, partially electroded piezoelectric actuators have the important advantage of the reduction of shear stress concentration and the related delamination problem over conventional fully electroded actuators. In [7], the substrate is an elastic plate modeled by classical two-dimensional plate equations. In [8], the plate substrate is governed by the three-dimensional equations of elasticity. However, in obtaining the electric potential distribution in the unelectroded portions of the piezoelectric film, an approximation is made [8] on the coupling of the potential to the elastic substrate; hence the problem is simplified and analytical solution is then possible. The solution in [8] is obtained by trigonometric series. Near the ends of an electrode, where the shear stress distribution changes more rapidly, the solution exhibits some oscillatory behavior [8], which resembles the Gibbs oscillation of a trigonometric series converging to a point of discontinuity. This oscillation is more pronounced in the trigonometric series solution of the shear stress distribution under a fully electroded piezoelectric actuator [5]. Since shear stress distribution under a partially electroded piezoelectric film is of fundamental importance to actuators, a more accurate analysis is desirable.

In this paper, the problem of shear stress distribution due to partially electroded piezoelectric actuators is formulated in a more fundamental way. We consider a partially electroded thin piezoelectric film on a semi-infinite elastic

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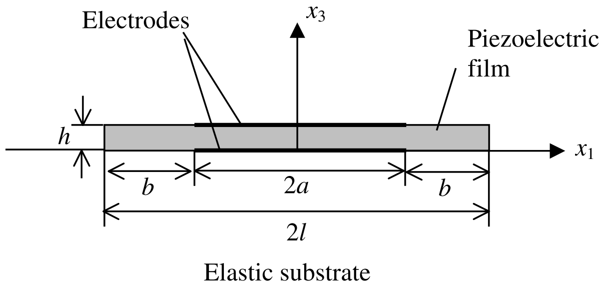

Figure 1. A partially electroded piezoelectric film on an elastic substrate.

substrate. The film is modeled by the equations for the extensional deformation of thin plates, and the substrate is governed by the equations of elasticity. The shear stress between the film and the substrate due to an applied voltage across the electroded portion of the film is shown to be controlled by a pair of coupled integro-differential equations. These equations are solved numerically using the concept of elements and the finite difference method, without the additional approximation introduced in [8], and the results are free from the undesirable oscillation as observed in the results of [5, 8]. The behavior of the shear stress is examined. The special cases of a fully electroded film on a rigid substrate, a partially electroded film on a rigid substrate, and a fully electroded film on a relatively soft substrate are solved analytically.

2. Formulation of the problem

Consider a partially electroded thin piezoelectric film on a semi-infinite elastic substrate as shown in figure 1. We study plane strain deformations with $u_2 = 0$ and $\partial/\partial x_2 = 0$. The top and bottom electrodes on the film are under a given applied electric potential $\pm V/2$. Then, for the electroded portion we have $\phi^{(0)} = 0$ and $\phi^{(1)} = V$, where $\phi^{(0)}$ and $\phi^{(1)}$ are the zero and first order plate electric potentials [6, 9]. In the unelectroded portions, since $\phi^{(0)}(\pm a) = 0$ and $\phi^{(0)}_{,1}(\pm l) = 0$, it can be concluded that $\phi^{(0)} = 0$. However, $\phi^{(1)}$ remains unknown in the unelectroded portions and is coupled to the mechanical fields [6, 7]. The shear stress between the film and the substrate is denoted by $\tau(x_1)$ for $|x_1| < l$. Since the film is assumed to be very thin, the continuity of the tangential displacement at the interface implies that the tangential surface displacement of the substrate is also $u_1^{(0)}(x_1)$. Then, from [11] we have the following relation for the surface strain due to a surface distribution of shear stress τ:

$$u^{(0)}_{1,1}(x_1) = -\frac{2(1-\nu^2)}{\pi E}\int_{-l}^{l}\frac{\tau(s)}{x_1 - s}\,\mathrm{d}s \tag{2.1}$$

where E is Young's modulus and ν is Poisson's ratio of the substrate. For the extensional force $T^{(0)}_{11}$ in the film we have [6, 7]:

$$T^{(0)}_{11} = hc^p_{11}u^{(0)}_{1,1} + e^p_{31}\phi^{(1)} \tag{2.2}$$

where c^p_{ij}, e^p_{ij} and ε^p_{ij} $(i, j = 1, 2, 3)$ are the material constants of the piezoelectric film [6, 7]. $T^{(0)}_{11}$ must satisfy the following edge conditions:

$$T^{(0)}_{11}(\pm l) = 0. \tag{2.3}$$

The equation of motion of the film reduces to [6, 7]:

$$\tau = T^{(0)}_{11,1}. \tag{2.4}$$

Integrating equation (2.4) from $-l$ to x_1, using equation (2.3), we obtain

$$\int_{-l}^{x_1}\tau(s)\,\mathrm{d}s = T^{(0)}_{11}(x_1). \tag{2.5}$$

Substituting equation (2.1) into equation (2.2), and the resulting expression into equation (2.5), we have the following integral equation for τ:

$$\int_{-l}^{x_1}\tau(s)\,\mathrm{d}s + \frac{hc^p_{11}2(1-\nu^2)}{\pi E}\int_{-l}^{l}\frac{\tau(s)}{x_1 - s}\,\mathrm{d}s = e^p_{31}\phi^{(1)}(x_1) \qquad |x_1| < l. \tag{2.6}$$

Formally, the same integral equation was obtained in [4], with different notations for the special case of a fully electroded piezoelectric film ($a = l$), but was not solved. The situation for a partially electroded film is more complicated in that we only have partial information of the electric potential

$$\phi^{(1)} = \begin{cases} V & |x_1| < a \\ \text{unknown} & a < |x_1| < l \end{cases} \tag{2.7}$$

where $\phi^{(1)}$ in the unelectroded portions is part of the unknowns and has to be obtained by solving simultaneous equations. The additional equation for determining $\phi^{(1)}$ [6, 7] takes the following form with the substitution of equation (2.1):

$$\frac{he^p_{31}2(1-\nu^2)}{\pi E}\int_{-1}^{l}\frac{\tau(s)}{x_1 - s}\,\mathrm{d}s = \frac{h^2}{12}\varepsilon^p_{11}\phi^{(1)}_{,11} - \varepsilon^p_{33}\phi^{(1)} \qquad a < |x_1| < l \tag{2.8}$$

which is coupled to τ. Equation (2.8) is an integro-differential equation, on which we need to impose the following boundary conditions [6, 7]

$$\phi^{(1)}(\pm a) = V \tag{2.9a}$$

$$\phi^{(1)}_{,1}(\pm l) = 0. \tag{2.9b}$$

The coupling to the mechanical fields in the equation for $\phi^{(1)}$, or effectively the term involving τ on the left-hand side of equation (2.8), was treated approximately in a different formulation in [8]. In summary, we have obtained a pair of coupled integro-differential equations (2.6) and (2.8) for determining $\tau(x_1)$ and $\phi^{(1)}(x_1)$, with boundary conditions in equation (2.9). We note that equation (2.8) is over the region of the unelectroded portions of the film only, where $\phi^{(1)}$ is unknown.

3. Some analytical results

3.1. A fully electroded film on a rigid substrate

For the special case of a rigid substrate we have $u^{(0)}_1 = 0$. Because of symmetry we only consider half of the film with $0 < x_1 < l$. For a fully electroded film with $a = l$, we have $\phi^{(1)} = V$ over the whole film. Equation (2.2) then implies that the film extensional force $T^{(0)}_{11}$ has a constant value of $e^p_{31}V$ along the film. Then, equation (2.4) further implies that the shear stress τ is zero under the film. However, equation (2.3) requires that $T_{11}^{(0)}$ must vanish at both ends of the film. This is possible only when two concentrated shear forces of equal magnitude $e_{31}^p V$ and opposite in direction are present at both ends of the film. Mathematically, the shear stress distribution can be represented by

$$\tau(x_1) = -e_{31}^p V[\delta(x_1 - l) - \delta(x_1 + l)] \qquad (3.1)$$

where $\delta(x_1)$ is the Dirac delta function.

3.2. A partially electroded film on a rigid substrate

In this case, from the equations in [6, 7], we have

$$e_{31}^p \phi_{,1}^{(1)} - \tau = 0 \qquad (3.2a)$$

$$-\frac{h^2}{12}\varepsilon_{11}^p \phi_{,11}^{(1)} + \varepsilon_{33}^p \phi^{(1)} = 0 \qquad (3.2b)$$

for which the boundary conditions in equations (2.9) still apply. From equations (3.2*b*) and (2.9) the following solution for $\phi^{(1)}$ can be obtained:

$$\phi^{(1)} = \begin{cases} V & 0 < x_1 < a \\ G\,\mathrm{e}^{-\xi(x_1-a)} + H\,\mathrm{e}^{\xi(x_1-a)} & a < x_1 < l \end{cases} \qquad (3.3)$$

where

$$\xi = \frac{2}{h}\sqrt{3\frac{\varepsilon_{33}^p}{\varepsilon_{11}^p}}$$

$$G = \frac{V}{1+\mathrm{e}^{-2\xi b}} \qquad H = \frac{V}{1+\mathrm{e}^{-2\xi b}}\,\mathrm{e}^{-2\xi b} \qquad (3.4)$$

and $b = l - a$, as shown in figure 1. The substitution of equation (3.3) into equation (3.2*a*) gives the shear stress distribution

$$\tau = \begin{cases} 0 & 0 < x_1 < a \\ \dfrac{e_{31}^p \xi V}{1+\mathrm{e}^{-2\xi b}}(-\mathrm{e}^{-\xi(x_1-a)} + \mathrm{e}^{-2\xi b}\,\mathrm{e}^{\xi(x_1-a)}) & \\ & a < x_1 < l \end{cases} \qquad (3.5)$$

We note that under the electroded portion there is no shear stress. In the unelectroded portion the shear stress decays exponentially with $\tau(l) = 0$. The important difference between equations (3.5) and (3.1) is that one predicts a distribution of finite shear stresses, while the other has a singular distribution of a delta function. This shows the advantage of partially electroded actuators in reducing stress concentration. The total shear force predicted by equation (3.5) can be obtained by integrating the shear stress over the unelectroded region (a, l), which yields

$$Q = \frac{e_{31}^p V}{1+\mathrm{e}^{-2\xi b}}(-1 + 2\,\mathrm{e}^{-\xi b} - \mathrm{e}^{-2\xi b}). \qquad (3.6)$$

We note that for the equations of a thin film to be valid in the unelectroded portion, the length of the unelectroded portion b has to be much larger than the thickness h. Then it can be seen from (3.4*a*) that in this case $\mathrm{e}^{-\xi b} \ll 1$. If

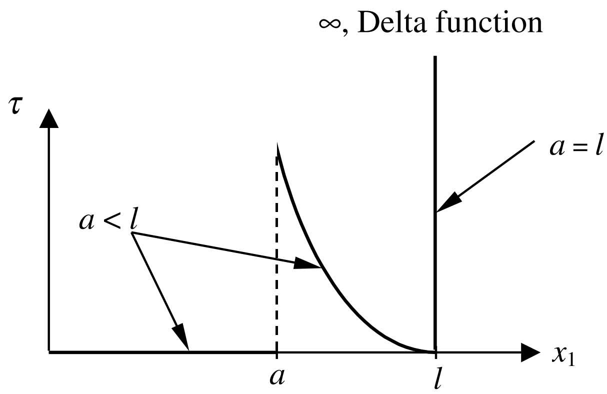

Figure 2. Shear stress distribution of a piezoelectric film on a rigid substrate.

we neglect $\mathrm{e}^{-\xi b}$ compared to 1 in the above expressions, we obtain the following physically more revealing expressions:

$$\phi^{(1)} \cong \begin{cases} V & 0 < x_1 < a \\ V\mathrm{e}^{-\xi(x_1-a)} & a < x_1 < l \end{cases} \qquad (3.7)$$

$$\tau \cong \begin{cases} 0 & 0 < x_1 < a \\ -e_{31}^p \xi V\,\mathrm{e}^{-\xi(x_1-a)} & a < x_1 < l \end{cases} \qquad (3.8)$$

$$Q \cong -e_{31}^p V. \qquad (3.9)$$

The shear stress distributions given by equations (3.1) and (3.8) are shown qualitatively in figure 2, where a minus sign has been dropped. It is seen that the distribution for a partially electroded film is much less concentrated than for a fully electroded one. From equation (3.9) it is seen that the total shear force is about the same as equation (3.1), being equal to the tension (or contraction) force in the unelectroded portion of the film.

3.3. A fully electroded film on a relatively soft substrate

We now consider the special case when the piezoelectric film is relatively much stiffer than the substrate. With the introduction of the following small dimensionless parameter

$$\lambda = \frac{E}{c_{11}^p 2(1-\nu^2)} \qquad (3.10)$$

and the change of the integration variable $s = lt$, equation (2.6) can be written as

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f(t)}{\varsigma - t}\,\mathrm{d}t = \lambda e_{31}^p \frac{V}{h} - \lambda\frac{l}{h}\int_{-1}^{\varsigma} f(t)\,\mathrm{d}t \qquad (3.11)$$

where we have denoted $f(t) = \tau(lt)$, and $\varsigma = x_1/l$. We now seek the following perturbation solution

$$f(t) = f^{(0)}(t) + \lambda f^{(1)}(t) + \cdots. \qquad (3.12)$$

Substituting equation (3.12) into equation (3.11), we obtain the following zero- and first-order problems:

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f^{(0)}(t)}{\varsigma - t}\,\mathrm{d}t = 0 \qquad (3.13a)$$

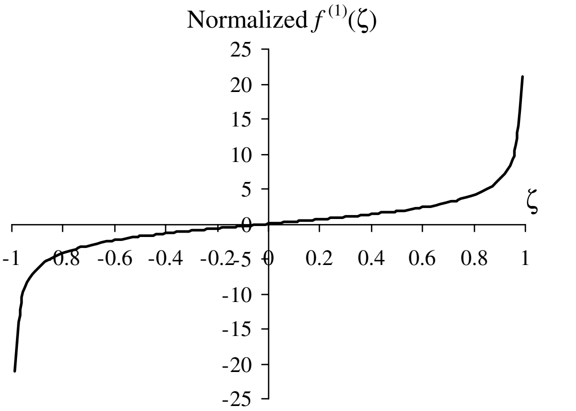

Figure 3. Normalized shear stress $\pi f^{(1)}(\varsigma)/(-e_{31}^p V/h)$.

$$\frac{1}{\pi}\int_{-1}^{1}\frac{f^{(1)}(t)}{\varsigma - t}\,\mathrm{d}t = e_{31}^p\frac{V}{h} - \frac{l}{h}\int_{-1}^{\varsigma} f^{(0)}(t)\,\mathrm{d}t. \qquad (3.13b)$$

These equations are analytically solvable [12, 13]. Equation (3.13a) is homogeneous and admits nontrivial solutions which are even functions. Since the shear stress distribution in our problem is a continuous odd function, the only physically meaningful solution of equation (3.13a) is $f^{(0)}(t) \equiv 0$, which represents the situation when the substrate is mechanically not resisting the deformation of the film. The first order solution from equation (3.13b) takes the form [12, 13]:

$$f^{(1)}(\varsigma) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-t^2}{1-\varsigma^2}}\frac{1}{\varsigma - t}\left(-e_{31}^p\frac{V}{h}\right)\mathrm{d}t \qquad (3.14)$$

or

$$\tau(x_1) \cong -\frac{E}{c_{11}^p 2(1-\nu^2)}e_{31}^p\frac{V}{h}\frac{1}{\pi}\int_{-l}^{l}\sqrt{\frac{l^2-s^2}{l^2-x_1^2}}\frac{1}{x_1 - s}\,\mathrm{d}s. \qquad (3.15)$$

It can be verified that equation (3.15) is an odd function and is singular at $x_1 = \pm l$. The integration of τ over $(0, l)$ is still finite to give the total shear force. $f^{(1)}(\varsigma)$ in equation (3.14) is normalized, and is shown in figure 3. Comparison of equations (3.15) and (3.1) shows that for the case of fully electroded actuators, the shear stress distribution is more concentrated when the substrate is stiffer than the film.

4. Numerical procedures

To solve the coupled integro-differential equations (2.6) and (2.8) for an elastic substrate, we introduce the concept of elements, which are used in the boundary element method (BEM) to solve boundary integral equations (see, e.g. [14, 15]). In addition, the finite difference method is also employed to deal with the derivative terms in the two integro-differential equations [16].

We start the numerical procedures by rewriting equations (2.6) and (2.8) as follows:

$$\int_{-l}^{x}\tau(s)\,\mathrm{d}s + A\int_{-l}^{l}\frac{\tau(s)}{x-s}\,\mathrm{d}s = e_{31}^p\Phi(x) \qquad |x| < l \quad (4.1)$$

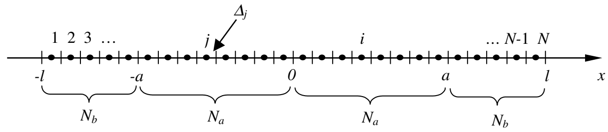

Figure 4. Discretization of the domain $(-l, l)$ into N $(=2N_a + 2N_b)$ small segments (elements).

and

$$B\int_{-l}^{l}\frac{\tau(s)}{x-s}\,\mathrm{d}s = C\Phi_{,11}(x) - \varepsilon_{33}^p\Phi(x) \qquad a < |x| < l \quad (4.2)$$

respectively, where

$$A = \frac{hc_{11}^p 2(1-\nu^2)}{\pi E} \qquad B = \frac{he_{31}^p 2(1-\nu^2)}{\pi E}$$

$$C = \frac{h^2}{12}\varepsilon_{11}^p \qquad \Phi = \phi^{(1)} \text{ and } x = x_1 \qquad (4.3)$$

have been introduced for simplicity. Then, we divide the whole interval $(-l, l)$ into small segments with length Δ_i, $i = 1, 2, 3, \ldots, N$, where N $(=2N_a+2N_b)$ is the total number of the segments, $2N_a$ is the number for the electroded portion and $2N_b$ is the number for the unelectroded portion (figure 4). One node is placed on each segment, which can be called an element. The shear stress and electrostatic potential are assumed to be constant over each element, which is the constant element as used in the boundary element method [15].

Let $x = x_i$ (i.e. at node i) for $i = 1, 2, 3, \ldots, N$; then the two integrals in equation (4.1) can be discretized as:

$$\int_{-l}^{x_i}\tau(s)\,\mathrm{d}s = \sum_{j=1}^{i}\int_{\Delta_j}\tau(s)\,\mathrm{d}s = \sum_{j=1}^{i}\tau_j\Delta_j \qquad (4.4)$$

$$\int_{-l}^{l}\frac{\tau(s)}{x_i - s}\,\mathrm{d}s = \sum_{j=1}^{N}\int_{\Delta_j}\frac{\tau(s)}{x_i - s}\,\mathrm{d}s = \sum_{j=1}^{N}\int_{\Delta_j}\frac{1}{x_i - s}\,\mathrm{d}s\,\tau_j$$
$$= \sum_{\substack{j=1\\ j\neq i}}^{N}\ln\left|\frac{1}{x_i - s}\right|_{x_j-\frac{1}{2}\Delta_j}^{x_j+\frac{1}{2}\Delta_j}\tau_j = \sum_{\substack{j=1\\ j\neq i}}^{N}\ln\left|\frac{x_i-(x_j-\frac{1}{2}\Delta_j)}{x_i-(x_j+\frac{1}{2}\Delta_j)}\right|\tau_j \qquad (4.5)$$

in which $\tau_j = \tau(x_i)$. The singular integral $\int_{\Delta_i}(x_i - s)^{-1}\,\mathrm{d}s$ can be shown to vanish in the Cauchy principal value (CPV) sense, and is thus excluded from the summation. Using the above two results, we can write the discretized form of equation (4.1) as

$$\Delta\sum_{j=1}^{i}\tau_j + A\sum_{\substack{j=1\\ j\neq i}}^{N}\ln\left|\frac{x_i-(x_j-\frac{1}{2}\Delta)}{x_i-(x_j+\frac{1}{2}\Delta)}\right|\tau_j = e_{31}^p\Phi_i$$
$$i = 1, 2, 3, \ldots, N \qquad (4.6)$$

where $\Phi_i = \Phi(x_i)$, and a uniform length for all the elements has been assumed (i.e. $\Delta_j = \Delta$, $j = 1, 2, 3, \ldots, N$). Notice that $\Phi_{N_b+1} = \Phi_{N_b+2} = \cdots = \Phi_{N_b+2N_a} = V$, as given in equation (2.7). There is a total of N equations in equation (4.6).

Applying the central difference scheme [16], we have

$$\Phi_{,11}(x_i) = \frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{\Delta^2}$$
$$i = 2, 3, \ldots, N_b, N_b + 2N_a + 1, \ldots, N-1. \qquad (4.7)$$

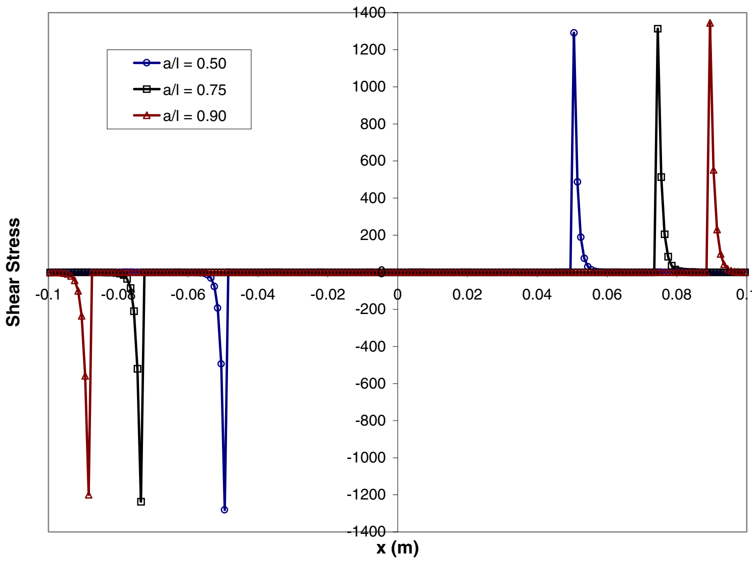

Figure 5. Shear stress distribution for different sizes of the electroded area.

Thus, the discretized form of equation (4.2) is found to be

$$B\sum_{\substack{j=1\\ j\neq i}}^{N} \ln\left|\frac{x_i-(x_j-\frac{1}{2}\Delta)}{x_i-(x_j+\frac{1}{2}\Delta)}\right|\tau_j = \frac{C}{\Delta^2}(\Phi_{i+1}-2\Phi_i+\Phi_{i-1}) - \varepsilon_{33}^{p}\Phi_i$$
$$i=2,3,\ldots,N_b,N_b+N_a+1,\ldots,N-1 \qquad (4.8)$$

in which

$$\Phi_{N_b+1}=\Phi_{N_b+2N_a}=V \qquad (4.9)$$

from condition (2.7) or (2.9*a*) should be applied. There are only $2N_b-2$ equations in equation (4.8). Two more equations are found by enforcing the zero-slope boundary condition in equation (2.9*b*) using the finite difference, that is:

$$\Phi_1=\Phi_2 \qquad \Phi_{N-1}=\Phi_N. \qquad (4.10)$$

There are a total of $N+2N_b$ equations in equations (4.6), (4.8) and (4.10), which are used to solve the N unknowns τ_i $(i = 1,2,3,\ldots,N)$ and the $2N_b$ unknowns Φ_i $(i = 1,2,3,\ldots,N_b,N_b+2N_a+1,\ldots,N-1,N)$. Equations (4.6), (4.8) and (4.10) can be written in a matrix form and solved for τ and Φ together, or solved for τ and Φ separately using partitioning of the whole matrix.

The symmetry of the problem can certainly be exploited in the discretization to reduce the size of the system of algebraic equations by half. However, the symmetry is not used in the discretization here since this is a 1-D, and hence less computationally intensive, problem. Rather, the symmetry feature is employed as an additional means in validating the numerical solutions of the coupled integro-differential equations.

5. Numerical results and discussions

As a numerical example, we consider PZT-7A piezo-electric films with the following material constants [17]:

$$\rho = 7500\ \text{kg m}^{-3} \qquad c_{11}=c_{22}=148$$
$$c_{33}=131 \qquad c_{12}=76.2 \qquad c_{13}=c_{23}=74.2$$
$$c_{44}=c_{55}=25.4 \qquad c_{66}=35.9\ \text{GPa}$$
$$e_{15}=9.2 \qquad e_{31}=-2.1 \qquad e_{33}=9.5\ \text{C m}^{-2}$$
$$\varepsilon_{11}=460\varepsilon_0 \qquad \varepsilon_{33}=235\varepsilon_0$$
$$\varepsilon_0=8.85\times10^{-12}\ \text{F m}^{-1}. \qquad (5.1)$$

For geometric parameters we choose $l = 0.1$ m and $h = 0.0025$ m. The substrate is made of steel ($E = 2.0\times10^{11}$ Pa, $\nu = 0.3$). Numerical tests show that $N = 200$ is sufficient and is thus used for all the following calculations.

Figures 5 and 6 show the results for the shear stresses and electric potentials respectively, with an electroded region of three different sizes. It is observed that the location of the peak value of shear stress moves towards the edges of the film when a increases, as expected. Since the shear stresses decay so rapidly, they essentially do not feel the ends of the film in the cases shown, with almost the same shear stress distributions concentrated at different locations. Figure 5 suggests that if a partially electroded actuator is used for the purpose of reducing the shear stress concentration, only very small portions in the order of about five times the film thickness near the ends of the actuator need to be left unelectroded. This provides some guidance for design.

From the numerical data of figure 5, it is observed that the shear stresses under the electrodes are not zero, but much smaller in values compared with the peak values near the edges of the electroded area. To illustrate this, the distributions of the shear stresses under the electrode are plotted in figure 7 for three substrates with decreasing Young's moduli. The shear stresses under the electrode are larger as the substrate becomes softer, as also suggested by figures 2 and 3. However, the values of the shear stress under the electrode are still several orders smaller than the overall

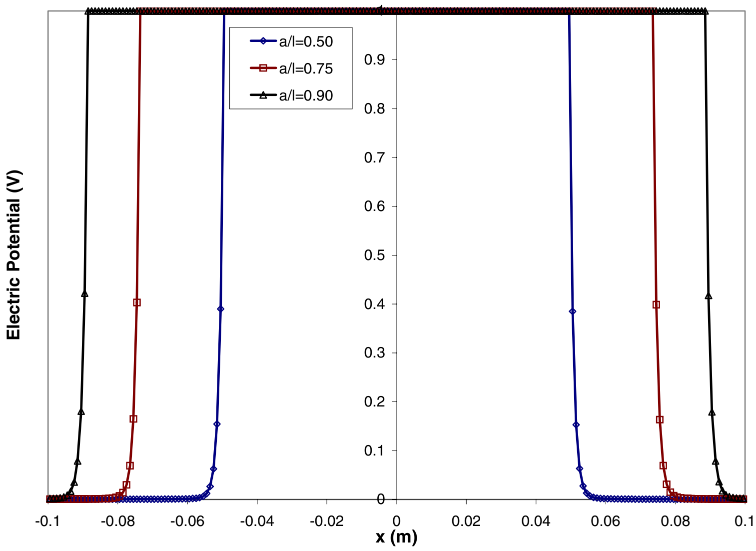

Figure 6. Electric potential distribution for different sizes of the electroded area.

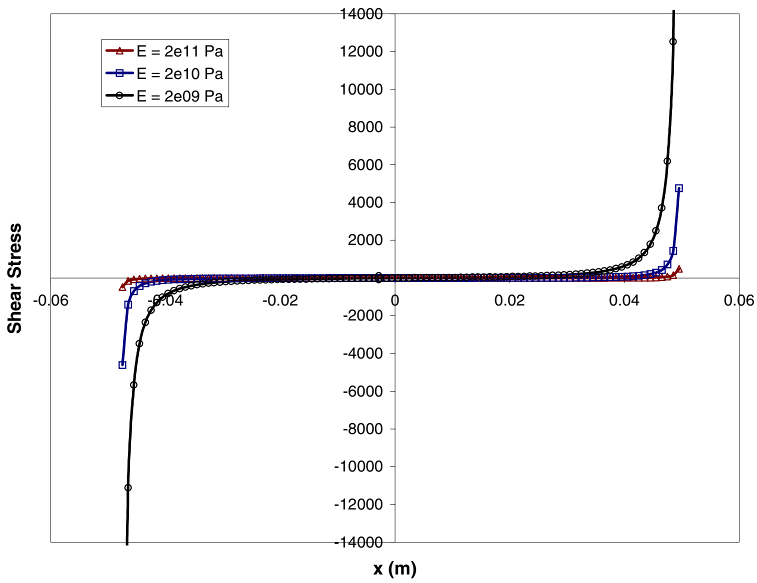

Figure 7. Distributions of the shear stresses ($\times 10^9$) under the electrodes.

peak values of the shear stress right outside the edges of the electrode. Practically, the shear stress under the electrode can be neglected for many purposes. Since the shear stress under the electrode is much smaller than the peak values right outside the electrode, the shear stress is discontinuous at the ends of the electrode. This is qualitatively different from the results in [8], where the shear stress is continuous everywhere. As pointed out in section 1, the results in [8] were obtained by Fourier series with some additional approximation, and suffer from the Gibbs oscillation near a discontinuity, which is the ends of the electrode as predicted in this paper. The shear stress predicted in [8] has to be continuous because of the method used, and as a result it can not describe a discontinuity and tends to overestimate the shear stress under the electrode. The discontinuity in the shear stress distribution is related to the thin film model used. It will be interesting to calculate this shear stress distribution from a fully three-dimensional model, which is left to future work.

6. Conclusions

Shear stress distribution under a partially electroded piezoelectric film is less concentrated than that of a fully electroded film. For a partially electroded film, the shear stress essentially concentrates under the unelectroded portions of the film, as long as the substrate is as stiff as or stiffer than the film. When the substrate is much softer than the film, more shear stress enters the electroded portion of the film. The case when the electrode ends are very close to the film edges requires a more refined analysis.

The numerical procedures developed in this paper have been shown to be very effective in solving the coupled integro-differential equations used for the studies. Very satisfactory numerical results are obtained using only the primitive constant elements. The numerical procedures can certainly be improved by employing higher order elements (e.g. linear or quadratic elements) to achieve higher accuracy and efficiency for the numerical solutions.

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