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A new form of the hypersingular boundary integral equation for 3-D acoustics and its implementation with C^0 boundary elements

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Abstract

An improved weakly-singular form of the hypersingular boundary integral equation (HBIE) for 3-D acoustic wave problems is presented in this paper. Compared with the weakly-singular form of the HBIE published earlier [Y.J. Liu and F.J. Rizzo, A weakly-singular form of the hypersingular boundary integral equation applied to 3-D acoustic wave problems, Comput. Methods Appl. Mech. Engrg. 96 (1992) 271–287], this new form involves only tangential derivatives of the density function and thus its discretization using the boundary element method (BEM) is easier to perform. Instead of using nonconforming and C^1 continuous boundary elements advocated earlier, C^0 continuous (conforming quadratic) elements are employed in the discretization of this weakly-singular form of the HBIE. Some justifications on using C^0 elements for HBIEs are provided to reflect the current views on this crucial issue for HBIEs. It is postulated that the original $C^{1,\alpha}$ continuity requirement for the density function in the analytical HBIE formulation can be relaxed to piecewise $C^{1,\alpha}$ continuity in the numerical implementation of the weakly-singular forms of the HBIE. Numerical examples of both scattering and radiation problems clearly demonstrate the accuracy and versatility of the new weakly-singular form of the HBIE for 3-D acoustics.

1. Introduction

Hypersingular boundary integral equations (HBIEs), which are derivatives of the conventional boundary integral equations (CBIEs), have become a useful alternative approach in the analysis of many mechanics problems, for which the CBIEs are insufficient or fail. In particular, for exterior acoustic or elastic wave problems in the frequency domain, the solutions of the CBIE formulations are nonunique at the eigenfrequencies of the associated *interior* problems [1,2]. These frequencies are called *fictitious eigenfrequencies* because they do not have any physical significance for the exterior problems. Burton and Miller's composite BIE formulation [3], using a linear combination of the CBIE and HBIE, has been demonstrated to be the most effective and theoretically-sound approach among all the methods available in dealing with this fictitious-eigenfrequency difficulty (FED) in acoustics [4–14]. An extension of this composite BIE formulation to elastodynamics [15,16] has also been shown to be effective in dealing with the FED in exterior *elastic* wave problems [17]. Most recently, it was found that the HBIEs can play a crucial part in the analysis of acoustic and elastic waves for thin bodies (thin structures, open cracks, etc.) [18–21]. Certainly, HBIEs will find more and more applications in the field of applied mechanics as their effectiveness and the weakly-singular features are recognized and employed in the BIE formulations and their BEM solutions.

The most difficult part in implementing HBIEs has been in dealing with the hypersingular integrals. Employing the weakly-singular or regularized forms of the HBIEs is, perhaps, one of the most desirable

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approaches to implement the HBIEs using the BEM. Various regularization procedures used to reduce the order of singularity in HBIEs have been devised in the last decade. A general introduction to HBIEs can be found in [22] and a comprehensive review on the regularization techniques in [23]. In the regularized forms of the HBIEs, the hypersingular integrals, which must be interpreted in the sense of Hadamard finite part (HFP) [24–26], are transformed to strongly singular integrals in the sense of Cauchy principal value (CPV) or better yet, weakly singular integrals for which the commonly used numerical procedures in the BEM are applicable. Many successful numerical computations of the HBIEs for various problems, employing the regularized forms of the HBIEs, have been reported in the references mentioned in the previous paragraph.

However, a serious theoretical issue associated with the HBIE formulations is the smoothness requirement. It has been shown [27,28] and gradually accepted in the BEM community [23] that for the hypersingular integral to exist, the derivatives of the density function must be Holder continuous (at least in the neighborhood of the source point). This means that *theoretically* only boundary elements that ensure the C^1 continuity near each node can be applied in the discretization of the HBIEs. These types of boundary elements include both the nonconforming elements where the nodes are moved inside the elements so that the continuity requirement is met in the neighborhood of the nodes [14,17,22,29,30], and the C^1 continuous elements such as the Overhauser and Hermite elements [14,17,31–35]. This stringent requirement for the density function in HBIEs has seriously hindered the applications and acceptance of the HBIEs in the BEM community, because of the inefficiency of the nonconforming elements and the complexity of the C^1 elements. Relaxation of this smoothness requirement on HBIEs has been attempted by several authors in the past using conforming boundary elements [12–14,36,37]. The validation of this relaxation received renewed attention recently [38,39] due to a strong desire to do so in the BEM community.

In the context of 3-D acoustics, the seeming difficulty in dealing with the hypersingular integrals and the associated issue about the efficiency of HBIEs have prevented the composite BIE from gaining popularity in engineering applications beyond the academic research. Earlier works on treatment of the hypersingular integrals in the Burton and Miller's BIE formulation give rise to various integrals which are still difficult to compute and usually limited to the use of constant boundary elements [4–11]. In 1990, Chien et al. [12] presented a comprehensive work on the composite BIE formulation for 3-D acoustics with a thorough review of the earlier works. In [12], the hypersingular integral in the HBIE was converted into a special Cauchy principal value integral and was evaluated using a special numerical procedure with adaptive subdivisions of the singular element. Most noteworthy in [12], quadratic C^0 (conforming) boundary elements, in violation of the theoretical smoothness requirement for HBIEs, were employed and, interestingly, very good numerical results were obtained. A similar approach, that is, to transform the hypersingular integral into a CPV integral before the discretization, was developed by Wu at el. in [13]. A more conservative approach was adopted in [13] where quadratic C^0 boundary elements were used for the discretization but with the collocation points, which are distinct from the nodes, located inside the elements in order to meet the smoothness requirement. A rectangular, overdetermined system matrix was obtained and a least-squares procedure was employed to solve this system. In 1992, Liu and Rizzo [14] presented a weakly-singular form of the hypersingular BIE for 3-D acoustic wave problems. In this weakly singular form of the HBIE, all the integrals are at most weakly singular and thus can be computed using the ordinary numerical quadrature in the BEM. Three types of boundary elements, namely, the conforming quadratic, nonconforming quadratic and Overhauser C^1 continuous elements, were employed in [14]. Very good numerical results for wavenumbers up to 5π were obtained by the nonconforming and Overhauser elements, both of which satisfy the smoothness requirement. The numerical results using conforming elements, which violate the smoothness requirement, did converge, but at a slower rate. Serious questions regarding the validity of using conforming boundary elements for HBIEs were raised in [14], including the overstated criticism of the work in [12] about the relaxation of the smoothness requirement.

It has been a dilemma for a long time that on one hand, theory dictates that smoothness requirement must be satisfied for the HBIEs to be meaningful; on the other hand, good numerical results have been obtained by using conforming C^0 boundary elements for various forms of the HBIEs. This unsettled situation for the HBIEs may be one of the main reasons for their slow acceptance in the BEM community, even though the formulation based on the HBIE has been proved to be a very sound and effective approach in overcoming the fictitious-eigenfrequency difficulty and thin-shape breakdown simultaneously in acoustic problems. In light of the recent thinking [38,39] on the smoothness requirement for HBIEs and its relaxation, and the continued successful studies using C^0 conforming elements for HBIEs [12–14,36,37], it is necessary to re-address the issue of smoothness and its relaxation for HBIEs, and clear the way for the applications of the HBIEs in various fields of mechanics.

In this paper, an improved weakly-singular form of the HBIE for 3-D acoustic wave problems is presented. Compared with the weakly-singular form of the HBIE for acoustic problems published earlier in [14], this new form involves only tangential derivatives of the density function, and thus its discretization using the BEM is easier to perform. Instead of using nonconforming and C^1 continuous boundary elements as advocated in [14], conforming quadratic elements are employed in the discretization of this new weakly-singular form of the HBIE. Justifications on relaxing the original smoothness (C^1 continuous) requirement on the HBIE are provided in this paper, in the context of acoustics, to reflect the current views on this long-debated issue about HBIEs. The new form of the HBIE is applied in the composite BIE formulation to overcome the fictitious eigenfrequency difficulties in 3-D acoustics using BIEs. Numerical examples of both scattering and radiation problems are given to demonstrate the accuracy and versatility of the new weakly-singular form of the HBIE.

2. The new weakly-singular form of the hypersingular BIE

We start with the following Helmholtz integral representation

$$C(P_0)\phi(P_0) = \int_S \left[G(P,P_0)\frac{\partial\phi(P)}{\partial n} - \frac{\partial G(P,P_0)}{\partial n}\phi(P) \right] \mathrm{d}S(P) + \phi^I(P_0)\,, \tag{1}$$

where ϕ is the total acoustic wave (the perturbation pressure or velocity potential) satisfying the Helmholtz equation $\nabla^2\phi + k^2\phi = 0$ for time harmonic waves, ϕ^I is a prescribed incident wave (for scattering problems only), $G(P,P_0) = e^{ikr}/(4\pi r)$ is the full space Green's function for the Helmholtz equation, and the coefficient $C(P_0) = 1$, $1/2$ or 0 when the source point P_0 is in the exterior region E (acoustic medium), on the boundary S (if it is smooth) or in the interior region B (a body or scatterer), respectively (Fig. 1). Eq. (1) with $P_0 \in S$ is the commonly used form of the conventional boundary integral equation (CBIE) for acoustic wave problems. This is a singular form of the CBIE which can be converted into a weakly-singular form readily using an integral expression for the coefficient $C(P_0)$ [40].

To derive the hypersingular BIE (HBIE), we first consider the following directional derivative of the representation integral (Eq. (1) with $P_0 \in E$) at point P_0 in the direction n_0 (n_0 will be the surface normal when P_0 is on the surface, Fig. 1)

$$\frac{\partial\phi(P_0)}{\partial n_0} = \int_S \left[\frac{\partial G(P,P_0)}{\partial n_0}\frac{\partial\phi(P)}{\partial n} - \frac{\partial^2 G(P,P_0)}{\partial n\,\partial n_0}\phi(P) \right] \mathrm{d}S(P) + \frac{\partial\phi^I(P_0)}{\partial n_0}\,, \quad \forall\, P_0 \in E\,. \tag{2}$$

When the source point P_0 approaches the boundary S, the second integrand becomes hypersingular (integrand is $O(1/r^3)$). In [14], a weakly-singular form of the HBIE was derived by employing a two-term Taylor's series subtraction from the density function and using the identities for the Green's function [41,42] to evaluate the added-back terms. In that subtraction, the total gradient of ϕ at the source point P_0 was used, that is,

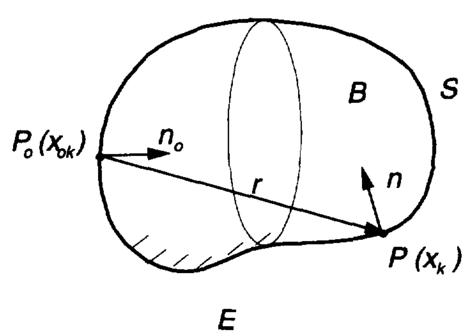

Fig. 1. The acoustic medium E, body B and boundary S.

$$\phi(P)-\phi(P_0)-\frac{\partial\phi(P_0)}{\partial x_k}(x_k-x_{0k}),\quad k=1,2,3.\quad \text{(summation over } k \text{ implied)}$$

Instead of using this total gradient, it is found that the subtraction using a 'surface gradient' or the tangential derivatives as expressed by

$$\phi(P)-\phi(P_0)-\frac{\partial\phi(P_0)}{\partial \xi_\alpha}(\xi_\alpha-\xi_{0\alpha}),\quad \alpha=1,2\quad \text{(summation over } \alpha \text{ implied)}$$

is sufficient to remove or regularize the hypersingularity of the kernel, where ξ_1 and ξ_2 are the first two (tangential) coordinates of a local curvilinear coordinate system $O\xi_1\xi_2\xi_3$ with origin at point P_0 (Fig. 2). This approach was adopted in [17] in regularizing the hypersingular BIE for elastodynamic problems.

To regularize the hypersingular and the strongly singular integrals in (2), we need the following three integral identities for the static Green's function $\bar{G}=1/(4\pi r)$ [41,42]:

$$\int_S \frac{\partial\bar{G}(P,P_0)}{\partial n}\,\mathrm{d}S(P)=0,\qquad \text{(the first identity)}$$

$$\int_S \frac{\partial^2\bar{G}(P,P_0)}{\partial n\,\partial n_0}\,\mathrm{d}S(P)=0,\qquad \text{(the second identity)}$$

$$\int_S \frac{\partial^2\bar{G}(P,P_0)}{\partial n\,\partial n_0}(x_k-x_{0k})\,\mathrm{d}S(P)=\int_S \frac{\partial\bar{G}(P,P_0)}{\partial n_0}n_k(P)\,\mathrm{d}S(P),\qquad \text{(the third identity)}$$

where the source point P_0 is in the domain E.

Following the procedure as described in [14], using the tangential derivatives in the subtraction, and employing the three integral identities shown above to evaluate the added-back terms, we can regularize the hypersingular integral in (2) as follows:

$$\begin{aligned}\int_S \frac{\partial^2 G(P,P_0)}{\partial n\,\partial n_0}\phi(P)\,\mathrm{d}S(P)&=\int_S \frac{\partial^2}{\partial n\,\partial n_0}[G(P,P_0)-\bar{G}(P,P_0)]\phi(p)\,\mathrm{d}S(P)\\&+\int_S \frac{\partial^2\bar{G}(P,P_0)}{\partial n\,\partial n_0}\left[\phi(p)-\phi(P_0)-\frac{\partial\phi(P_0)}{\partial\xi_\alpha}(\xi_\alpha-\xi_{0\alpha})\right]\mathrm{d}S(P)\\&+e_{\alpha k}\frac{\partial\phi(P_0)}{\partial\xi_\alpha}\int_S\left[\frac{\partial\bar{G}(P,P_0)}{\partial n_0}n_k(P)+\frac{\partial\bar{G}(P,P_0)}{\partial n}n_k(P_0)\right]\mathrm{d}S(P),\quad \forall P_0\in E,\end{aligned}\tag{3}$$

where $e_{\alpha k}=\partial\xi_\alpha/\partial x_k$ $(k=1,2,3)$ are the first two column vectors of the inverse of the Jacobian matrix and $(\xi_\alpha-\xi_{0\alpha})=e_{\alpha k}(x_k-x_{0k})$. The singular integral in (2) can also be regularized using the first integral identity shown previously to yield:

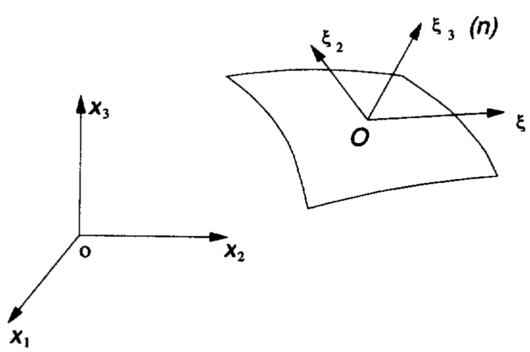

Fig. 2. The global and local coordinate systems.

$$\int_S \frac{\partial G(P,P_0)}{\partial n_0}\frac{\partial \phi(P)}{\partial n}\,\mathrm{d}S(P) = \int_S \left[\frac{\partial G(P,P_0)}{\partial n_0} + \frac{\partial \bar{G}(P,P_0)}{\partial n}\right]\frac{\partial \phi(P)}{\partial n}\,\mathrm{d}S(P) - \int_S \frac{\partial \bar{G}(P,P_0)}{\partial n}\left[\frac{\partial \phi(P)}{\partial n} - \frac{\partial \phi(P_0)}{\partial n}\right]\mathrm{d}S(P)\,, \quad \forall P_0 \in E\,, \tag{4}$$

where

$$\left[\frac{\partial G(P,P_0)}{\partial n_0} + \frac{\partial \bar{G}(P,P_0)}{\partial n}\right] = \left[\frac{\partial G(P,P_0)}{\partial n_0} - \frac{\partial \bar{G}(P,P_0)}{\partial n_0}\right] + \frac{\partial \bar{G}(P,P_0)}{\partial x_k}[n_k(P) - n_k(P_0)]$$

which represents a weakly-singular kernel as P approaches P_0 when S is smooth near P_0.

Substituting (3) and (4) into (2) and letting $P_0 \to S$, we obtain the new weakly-singular form of the hypersingular BIE:

$$\begin{aligned}
&\frac{\partial \phi(P_0)}{\partial n_0} + \int_S \frac{\partial^2 \bar{G}(P,P_0)}{\partial n\,\partial n_0}\left[\phi(P) - \phi(P_0) - \frac{\partial \phi(P_0)}{\partial \xi_\alpha}(\xi_\alpha - \xi_{0\alpha})\right]\mathrm{d}S(P) \\
&+ \int_S \frac{\partial^2}{\partial n\,\partial n_0}[G(P,P_0) - \bar{G}(P,P_0)]\phi(P)\,\mathrm{d}S(P) \\
&+ e_{\alpha k}\frac{\partial \phi(P_0)}{\partial \xi_\alpha}\int_S \left[\frac{\partial \bar{G}(P,P_0)}{\partial n_0}n_k(P) + \frac{\partial \bar{G}(P,P_0)}{\partial n}n_k(P_0)\right]\mathrm{d}S(P) \\
&= \int_S \left[\frac{\partial G(P,P_0)}{\partial n_0} + \frac{\partial \bar{G}(P,P_0)}{\partial n}\right]\frac{\partial \phi(P)}{\partial n}\,\mathrm{d}S(P) \\
&- \int_S \frac{\partial \bar{G}(P,P_0)}{\partial n}\left[\frac{\partial \phi(P)}{\partial n} - \frac{\partial \phi(P_0)}{\partial n}\right]\mathrm{d}S(P) + \frac{\partial \phi^I(P_0)}{\partial n_0}\,, \quad \forall P_0 \in S\,,
\end{aligned} \tag{5}$$

in which all the integrals are at most weakly-singular, *if $\phi(P)$ has continuous first derivatives*. This theoretical continuity requirement on the density function $\phi(P)$ and its relaxation in the discretization of the HBIE (5) will be further discussed in the next section.

Eq. (5) is the desired weakly-singular form of the hypersingular BIE for acoustic wave problems. It is interesting to note that the HBIE for acoustics in the form of (5) exhibits a term-by-term correspondence with the HBIE for elastodynamics developed in [17]. Compared with the form used earlier [14], this new form is much easier to discretize because the tangential derivatives of $\phi(P)$ can be evaluated readily using shape functions on an element. The discretization procedure for (5) is similar to that described in [14].

The well-known Burton and Miller's formulation [3], or composite BIE, using a linear combination of the CBIE and HBIE as represented symbolically by

$$\text{CBIE} + \beta\text{HBIE} = 0$$

with β being a coupling constant, can furnish unique solutions at all frequencies. Discussions on the choices of β, which have been found to be rather non-restrict, are given in [5,12,14,43,44]. This composite BIE has been demonstrated to be the most effective and theoretically-sound approach in acoustics and sound-structure interaction problems [12–14,45], as long as the hypersingular integral is dealt with properly.

3. Relaxation of the smoothness requirement for the HBIE

It has been well established that the theory imposes a $C^{1,\alpha}$ continuity requirement on the density function $\phi(P)$, in order for the following limit of the hypersingular integral

$$\lim_{P_0 \to S}\int_S \frac{\partial^2 \bar{G}(P,P_0)}{\partial n\,\partial n_0}\phi(P)\,\mathrm{d}S(P) \tag{6}$$

to exist [25,27–29]. This means that the derivatives of the density function $\phi(P)$ must be Holder continuous in the neighborhood of the source point P_0. This will exclude, *theoretically*, the use of C^0 boundary elements, such as the conforming quadratic elements, in the discretizations of hypersingular BIEs.

In the previous numerical work [14,17,20,21,29,30], this smoothness requirement was enforced to the full compliance. The nonconforming quadratic elements, which are formed by moving the nodes inside the elements so that the smoothness requirement is met in the neighborhood of the nodes, were applied for HBIEs in [14,29]. The Overhauser C^1 continuous surface elements [31,32] were applied in [14,33]. These two types of boundary elements have been used successfully for the implementations of the hypersingular BIEs for various validation type of problems, but they suffer serious drawbacks in real applications regarding the efficiency. The nonconforming elements are not even C^0 continuous across the element boundaries. Furthermore, since the nodes are no longer shared by neighboring elements, more nodes and hence a larger system of linear equations will be needed if the same number of elements is used as for the conforming elements. The Overhauser C^1 (quadrilateral and triangular) elements are very accurate, as demonstrated for 3-D acoustic and elastic wave problems in [14,17,33]. However, their implementations are difficult for 3-D problems due to their complexity, especially for 3-D domains with edges and corners. The Overhauser elements for 2-D problems with corners have been successfully implemented in [35]. A recent extension of the Overhauser elements to construct C^2 continuous boundary elements for 2-D problems can be found in [46]. However, the gains in accuracy by employing the higher order elements (beyond C^0 quadratic elements) are often offset by the lost in efficiency in BEM discretizations, which is even more so for 3-D problems.

Adopting a different approach to avoid the use of nonconforming or C^1 continuous elements, several authors have attempted using C^0 conforming quadratic elements for various regularized forms of the HBIEs and obtained good numerical results. In [12], the hypersingular integral in the HBIE was converted into a special Cauchy principal value integral and evaluated using a special numerical procedure with adaptive subdivisions of the singular element. C^0 conforming quadratic boundary elements, in violation of the theoretical smoothness requirement for HBIEs, were employed and, interestingly enough, very good numerical results were obtained. A similar approach to transform the hypersingular integral into a CPV integral before the discretization was developed in [13]. A more conservative method was adopted in [13] where quadratic C^0 boundary elements were used for the discretization, but with the collocation points, which are distinct from the nodes, being placed inside the elements. In [14] the conforming quadratic elements were also applied for a weakly-singular form of the HBIE for 3-D acoustic wave problems, in addition to the nonconforming and Overhauser elements used. The numerical results using conforming quadratic elements did converge, though at a slower rate. All these successful attempts, as well as the tests for elastostatic problems [37,38,47], have shown that the implementation of the C^0 conforming elements for HBIEs is possible and efficient. The question is *not* 'if this can be done', but 'why this can be done' and how to justify this approach. In the following, justifications for using the C^0 conforming elements for HBIEs are provided, based on an analogy of the BIE formulations with the finite element method (FEM) regarding the strong and weak formulations.

It is well known in the FEM that for a given boundary-value problem, one can start with either a *strong* formulation or a *weak* formulation to perform the finite element discretizations. Strong formulation and weak formulation have different requirements on the smoothness of the functions involved. For example, consider the following classical two-point boundary-value problem (BVP) (see e.g. [48]):

$$-u''(x) + u(x) = 0\,, \quad x \in (0, 1)$$
$$u(0) = 0\,, \qquad u'(1) = a$$

where a is a given constant. One can start with the following statement:

$$\int_0^1 (-u'' + u)v \, dx = 0 \tag{7}$$

where v is the test function in a function space. In (7), it is necessary for $u''(x)$ to be continuous for the integral to be meaningful. Applying integration by parts for (7), one arrives at

$$\int_0^1 (u'v' + uv) \, dx = av(1)\,. \tag{8}$$

Here, it is sufficient for $u'(x)$ to be continuous. Statement (7) is a weighted residual or *strong* formulation, while statement (8) a variational or *weak* formulation, for the original BVP. They have different regularity or smoothness requirements for the functions involved in order for the integrals to be meaningful. Most of the FEM formulations are based on the weak or variational formulations such as the one shown in (8), since the smoothness requirement for the weak formulations is less than that for the original BVP or the strong formulations.

It is argued here that the HBIEs containing the hypersingular integrals in the sense of HFP, as expressed by (6), are similar to the strong statement in (7); while the weakly-singular forms of the HBIEs containing weakly-singular integrals, as expressed by

$$\int_S \frac{\partial^2 \overline{G}(P,P_0)}{\partial n\,\partial n_0}\left[\phi(P)-\phi(P_0)-\frac{\partial\phi(P_0)}{\partial\xi_\alpha}(\xi_\alpha-\xi_{0\alpha})\right]\mathrm{d}S(P)\,, \tag{9}$$

are similar to the weak statement in (8). However, for the BIE formulations it is the *order of singularity* of the integrand that is strong or weak; while for the FEM formulations it is the *order of derivatives* in the integrand that leads to the phrase strong or weak. If this analogy using the strong and weak formulations between the FEM and BIE holds, then we can have a clearer picture of the smoothness issue for the HBIE formulations. Similar to the FEM formulation, it is natural to have a stronger smoothness requirement for the function in the strong formulation (expression (6) in the sense of HFP), since it involves higher order of singularities. This stronger smoothness requirement should be assumed as well in the derivations of the weak formulation from the strong one. However, once the weak formulation (expression (9)) is obtained, one can relax the smoothness requirement for the function involved, based on the new conditions under which valid *numerical* solutions can be obtained from the new formulation, which has been the case in the FEM (cf. Eqs. (7) and (8)). The ultimate question will be: will the numerical solution *converge* to the analytical solution? Although it is difficult to provide, and it may not appear for sometime, the proof of the convergence for the weakly-singular forms of the HBIEs with conforming elements, one should not cease to apply these weak forms of HBIEs with conforming elements. Numerical tests on the convergence can be carried out easily and they often provide reliable indications of the behaviors of the formulations.

Then, what is the condition under which valid numerical solutions using the weakly-singular form of the HBIE can be obtained? For the purpose of discussion, let us assume that S in the integral (9) is composed of two boundary elements and the source point P_0 is at the mid-node on the common edge of the two elements. Examining the weakly-singular integral in (9) carefully, one notices that the integral will be meaningful, i.e. provide a finite number (in fact, the sum of two separate finite numbers from each element), if the density function ϕ is *piecewise* $C^{1,\alpha}$ continuous over each individual element. This means that C^0 conforming quadratic elements, which are piecewise $C^{1,\alpha}$ continuous as well, can be applied to the weakly-singular forms of the HBIEs. This is exactly the same argument used in the FEM for the weak formulation (8) where less smooth functions can be used, because of the specific form of the integral present in that formulation.

From the above discussions, we postulate that the original $C^{1,\alpha}$ continuity requirement on the density function in the *analytical* HBIE formulation can be relaxed to *piecewise* $C^{1,\alpha}$ continuity in the *numerical* implementation of the *weakly-singular forms* of the HBIE. This relaxation means that conforming linear, quadratic, and other higher-order elements, as well as nonconforming elements (including the constant elements), can be applied to the weakly-singular forms of the HBIEs. With this relaxation, we can also avoid the awkward situation, associated with the applications of HBIEs for a long time, that nonconforming elements (which are not even C^0 continuous) are advocated to be applied for HBIEs, while conforming elements (which are at least C^0 continuous) are prohibited from being applied to HBIEs, according to the original $C^{1,\alpha}$ continuity requirement. The focus on the smoothness issues should now be turned to proving the convergence of the weakly-singular forms of the HBIEs with conforming elements. The theoretical proof may be hard to attain, but the numerical tests, including the ones provided in the next section and by others [12,37,38,47], can provide a clear indication about the validity of this relaxation.

4. Numerical examples

Studies on the scattering and radiation from spherical and cylindrical bodies were conducted to verify the developed composite BIE with the conforming quadratic elements (Fig. 3).

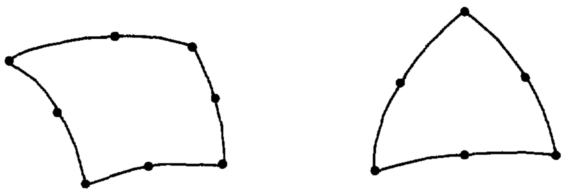

Fig. 3. Conforming quadratic boundary elements.

The first numerical study is for a spherical body (of radius a) immersed in an acoustic medium (Fig. 4), for which analytical solutions [12,49] are available for both the scattering and radiation problems. In all cases, M is the total number of elements used.

Fig. 5 shows the scattered wave at $ka = \pi$ when the sphere is impinged upon by an incident wave ϕ^I in the x_3-direction, using the composite BIE with conforming quadratic elements. The sphere is rigid where $\partial\phi/\partial n = 0$ on S. Magnitude of the normalized scattered wave $|\phi^S/\phi^I|$ at $r = 5a$ is plotted versus the angle θ (Fig. 4). Wavenumber $ka = \pi$ is a fictitious eigenfrequency for the conventional BIE, so CBIE can not be applied successfully (condition numbers of the system of equations are in the range of $10^6 \sim 10^7$, a clear indication of the non-uniqueness of the CBIE solutions). However, the results using the composite BIE are very stable and low condition numbers (below a few hundreds) are observed in all the cases. Fig. 5 clearly demonstrates the convergence of the results using the conforming quadratic elements for the composite BIE.

Fig. 6 shows the radiated waves when the sphere is pulsating with a uniform radial velocity v_0 on the surface S and $\partial\phi/\partial n = ikz_0v_0$, with z_0 being the characteristic impedance. The magnitude of the normalized surface pressure is plotted for wavenumbers $ka = 0$ to 7 with an increment of 0.1. With this small frequency increment, the fictitious eigenfrequencies for the CBIE can be identified clearly at $ka = \pi$ and 2π, near which the CBIE results deviate substantially from the analytical solution. The composite BIE, however, provides very satisfactory and stable results throughout the range of the frequencies. Composite BIE results between $ka = 4 \sim 6$ are not as good as those using the CBIE, but can be improved significantly with a finer mesh (results are not shown on the plot). The data (at 71 frequencies) for the composite BIE using 80 elements were obtained in less than 10 minutes on a PC with a Pentium Pro 200 MHz processor and 64 Mb RAM.

The second study is for a cylindrical (capsule-like) body with radius = 1.0 m and total length = 7.0 m. Since no analytical solutions are readily available for this problem, the commercial boundary element software *COMET/Acoustics* is employed in the verification for the radiation problem. Analysis of scattering problems is not available in *COMET/Acoustics*. The same mesh with 216 elements and 626 nodes (Fig. 7) is used for both *COMET/Acoustics* and the developed composite BIE code.

For the scattering problem, the capsule is impinged upon by an incident wave ϕ^I in the x-direction (Fig. 7).

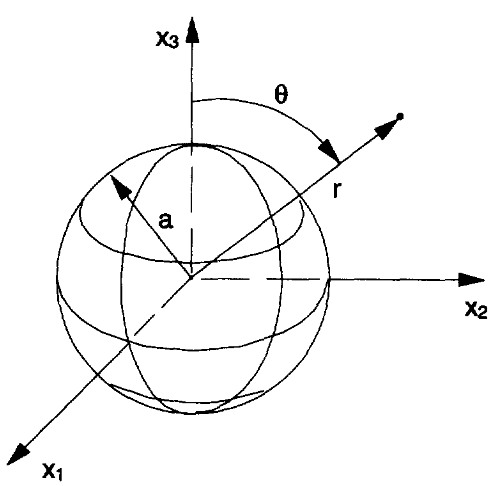

Fig. 4. A spherical body.

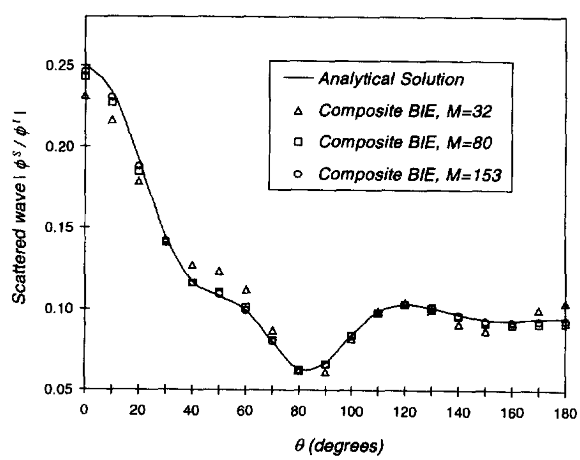

Fig. 5. Scattering from the rigid sphere at $r = 5a$, $ka = \pi$; composite BIE ($\beta = 0.3i$).

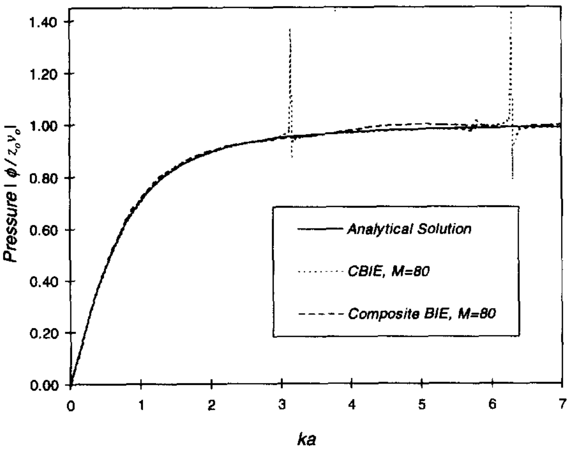

Fig. 6. Pressure on the surface of the pulsating sphere.

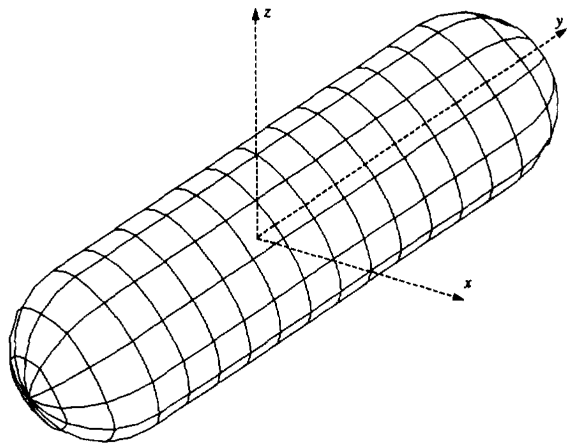

Fig. 7. A cylindrical (capsule-like) body with radius = 1.0 m and total length = 7.0 m.

Scattered waves for frequencies from 0 to 250 Hz (with 100 frequency steps), at the two points $(-10, 0, 0)$ (backscattering) and $(0, 10, 0)$ in the main axis direction, are plotted in Figs. 8 and 9, respectively. Four fictitious eigenfrequencies of the CBIE near 134, 153, 185 and 224 Hz were identified by monitoring the condition number of the system of equations at each frequency and the stability in the CBIE results. The composite BIE provided stable and smooth results throughout the frequency range, with very low condition numbers observed.

For the radiation problem (a pulsating capsule), a uniform normal velocity of unit magnitude is applied on the whole surface of the cylinder. Radiated waves for frequencies from 0 to 250 Hz (with 100 frequency steps), at the two points $(10, 0, 0)$ in the lateral direction and $(0, 10, 0)$ in the axis direction, are plotted in Figs. 10 and 11, respectively. *COMET* direct BIE is based on the same conventional BIE formulation as the one used in this research, and suffers from the same fictitious eigenfrequency difficulty (FED). The CHIEF method [1,50] is used in *COMET/Acoustics* to overcome this difficulty. The CHIEF method uses the Helmhotz integral representation at additional points inside the body (interior points) and solves a over-determined system of equations [1,50]. It is well known that the success of the CHIEF method to overcome the FED is largely determined by successful selections of the interior points, which are case-dependent and often difficult for complicated structures. In this case, six interior points were placed along the axis of the cylinder for the *COMET* CHIEF solution (one or two points were found to be insufficient). Figs. 10 and 11 show that both the results using *COMET* direct BIE and the CBIE deteriorate near the four fictitious eigenfrequencies (134, 153, 185 and

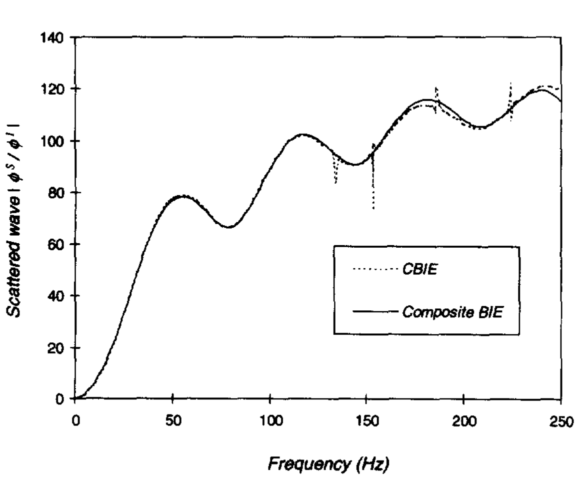

Fig. 8. Backscattering from the cylinder (with side incident wave).

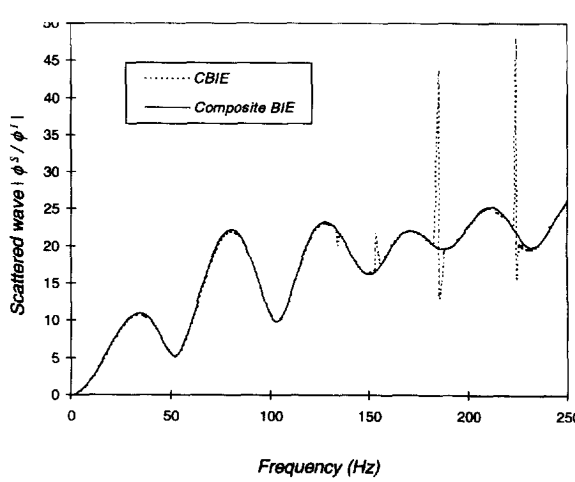

Fig. 9. Scattering from the cylinder in the main axis direction.

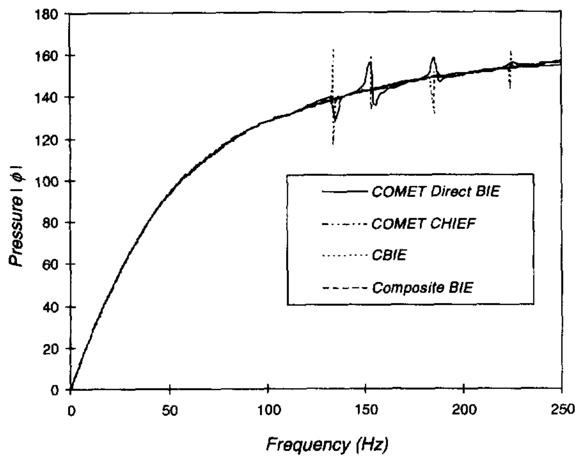

Fig. 10. Radiated wave from the cylinder in the lateral direction.

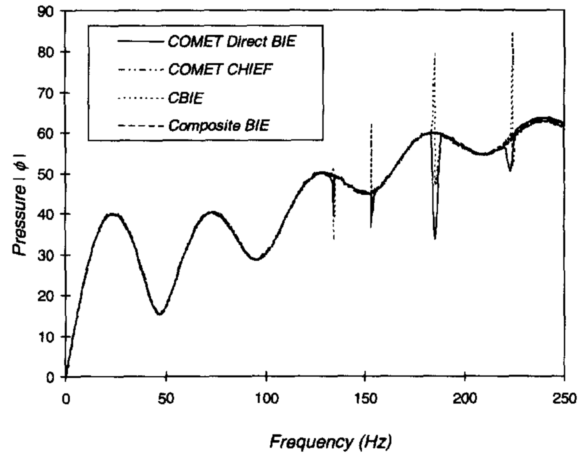

Fig. 11. Radiated wave from the cylinder in the main axis direction.

224 Hz), while the results using *COMET* CHIEF and the composite BIE stay closely along a smooth curve, as expected.

5. Conclusion

A new weakly-singular form, containing only the tangential derivatives of the density function, of the hypersingular boundary integral equation for 3-D acoustic wave problems is presented in this paper. Conforming C^0 quadratic boundary elements are employed in the discretization which is quite straightforward and does not require any special numerical integration schemes beyond that used for the conventional BIE. Some justifications on using C^0 elements for HBIEs are provided to reflect the current views on this crucial issue. It is postulated that the original continuity requirement for the density function in the analytical HBIE formulation can be relaxed to *piecewise* $C^{1,\alpha}$ continuity in the *numerical* implementation of the *weakly-singular forms* of the HBIE. Numerical examples, ranging from scattering and radiation problems from different geometries, clearly demonstrate the effectiveness and efficiency of the improved composite BIE approach to 3-D acoustic problems.

Investigations on the composite BIE formulation for thin bodies with non-zero thickness, the interaction of sound and shell-like structures, and the corner problem for the HBIE using conforming elements are underway and will be reported subsequently.

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