Scattering of elastic waves from thin shapes in three dimensions using the composite boundary integral equation formulation

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In this paper, the composite boundary integral equation (BIE) formulation is applied to scattering of elastic waves from thin shapes with small but *finite* thickness (open cracks or thin voids, thin inclusions, thin-layer interfaces, etc.), which are modeled with *two surfaces*. This composite BIE formulation, which is an extension of the Burton and Miller's formulation for acoustic waves, uses a linear combination of the conventional BIE and the hypersingular BIE. For thin shapes, the conventional BIE, as well as the hypersingular BIE, will degenerate (or nearly degenerate) if they are applied *individually* on the two surfaces. The composite BIE formulation, however, will not degenerate for such problems, as demonstrated in this paper. Nearly singular and hypersingular integrals, which arise in problems involving thin shapes modeled with two surfaces, are transformed into sums of weakly singular integrals and nonsingular line integrals. Thus, no finer mesh is needed to compute these nearly singular integrals. Numerical examples of elastic waves scattered from penny-shaped cracks with varying openings are presented to demonstrate the effectiveness of the composite BIE formulation. © *1997 Acoustical Society of America.* [S0001-4966(97)05008-X]

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INTRODUCTION

The modeling of thin shapes or thin bodies (including shell-like structures, thin inclusions, thin voids or open cracks in solids, thin-layer interfaces in composites, etc.) is of increasing importance and interest in the fields of acoustics, elastodynamics, and fluid-structure interactions. It is well known that the conventional boundary integral equation (CBIE) formulations, as well as the hypersingular BIE (HBIE) formulations for acoustic and elastic wave problems, will degenerate or break down when they are applied to thin bodies.[1–3] This degeneracy or breakdown is due to the fact that the equation on one side of the thin shape is almost the same as the equation on the other side, see Fig. 1. Eventually the two equations from the two sides become identical when the thickness of the thin shape approaches zero. In the literature, the BIE formulations developed for dealing with the thin shape breakdown can be divided into two groups: one applied to the *one surface model* and one to the *two surface model* of the thin shape. One exception to this classification is the multidomain method[4,5] which can be used for both one surface and two surface models.

The one surface model of a thin shape is an idealized model in which the middle surface of the thin shape is chosen and usually the hypersingular BIE is applied in terms of the pressure jump (for acoustic problems) or the displacement jump (for elastodynamic problems) across the thin shape. This approach has been applied successfully to the problems of scattering and radiation of *acoustic* waves from thin rigid bodies, see, e.g., Refs. 6–11. The scattering of *elastic* waves from planar cracks in three-dimensional elastic medium is also studied using this one surface approach.[7,12,13] This single surface model for thin shapes is efficient in modeling and computation, as long as the hypersingular integrals in the BIE formulations are dealt with properly. However, there are some drawbacks and limitations with this approach. For example, effects of the varying thickness of a thin body or the opening of a crack cannot be studied using this simple model. The use of the jump terms (e.g., pressure jump), instead of the usual boundary variables (pressure), in the hypersingular BIE may also present some inconveniences in the study of a regular body with thin shapes, where the regular BIE and the hypersingular BIE for thin shapes need to be used together.[10]

The two surface model of a thin shape is a more realistic model in that the geometry of the thin shape is not altered. The effects of the thinness and other details of the thin shape can be addressed easily using the two surface model. Early studies of scattering of *acoustic* waves from thin shapes (rigid disks and fluid inclusions with varying thickness in acoustic media) can be found in Refs. 2, 3, and 14. In these studies, the conventional BIE is used on one surface of the thin shape and the hypersingular BIE on the other surface to obtain a nondegenerate system of equations, see Fig. 2. The use of the two surface model is more computation intensive than the one surface model, but it does provide more information about the physics of the problem than the one surface model, as demonstrated in these studies. For example, in the case of acoustic scattering from a rigid disk, the scattered fields from the disk of a finite thickness $2h$ will depart from those fields from the disk of zero thickness, when h is greater than 5% of the radius of the disk. This difference is more

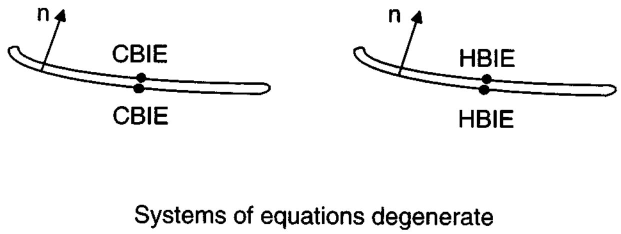

FIG. 1. Degeneracy of the CBIE and HBIE for thin shapes.

pronounced in the near field or when the incident wave is in the direction parallel to the surfaces of the disk, especially if the disk is a ''rippled'' one.[2,3] All these studies on acoustic waves from thin shapes show that the two surface model using the combination of CBIE and HBIE can provide a general BIE formulation and useful information in the analysis of thin shapes. The same BIE formulation can be applied to thin shapes with zero thickness or small thickness, as well as to bulky bodies.

The multidomain method[4,5] is simple and straightforward. It can be applied to thin shapes with zero thickness or nonzero thickness. In this method, the difficulty of dealing with the degeneracy of the conventional BIE for thin shapes is avoided by introducing an imaginary interface to divide the domain into an interior subdomain and an exterior subdomain. The conventional BIE is applied in the two subdomains with meshes on the thin body and interface surfaces, and the continuity conditions are imposed on the interface. This results in a larger system of equations and hence increases the burden of computation, because of the introduction of the imaginary interface. The multidomain method has been used effectively for problems of radiation and scattering of *acoustic* waves from thin rigid bodies.[4,5] Although this approach may be applied to thin body problems posed in a more general setting (e.g., with varying thickness), ''it may not be an ideal tool for the thin-body radiation and scattering problem in which a relatively large imaginary interface surface is usually required.'' [9]

In this paper, the idea of using a combination of the conventional BIE and the hypersingular BIE for thin shapes[2,3,14] is extended to the problems of scattering and radiation of *elastic waves* from thin shapes in elastic media. An alternative form of the CBIE and HBIE combination is used here for the thin shapes, that is, a linear combination of the CBIE and HBIE applied on both surfaces of the thin shapes, see Fig. 2. This composite BIE formulation for elastic wave problems is an extension of the well-known Burton and Miller's BIE formulation[15] for acoustic wave problems, which can remove the fictitious eigenfrequency difficulties in exterior problems. It has also been shown analytically,[3] and will be demonstrated numerically in this paper, that this composite BIE formulation will not degenerate when applied to thin shapes modeled with two distinct surfaces. Thus Burton and Miller's type of BIE formulation can overcome the fictitious eigenfrequency difficulty and the thin body break down difficulty at the same time, being likely the most sound and robust BIE formulations for the acoustic and elastic wave problems.

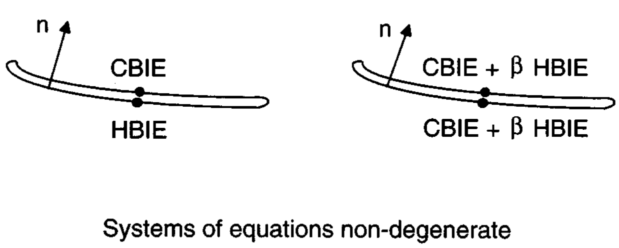

FIG. 2. Nondegeneracy of the combinations of the CBIE and HBIE for thin shapes.

For elastic wave problems, the conventional BIE formulation, which contains strongly singular kernels, has been applied successfully to *bulky-shaped* voids or inclusions for almost a decade (see, e.g., Refs. 16–19). On the other hand, the hypersingular BIE, which contains hypersingular kernels and usually written on one surface of a crack and in terms of the crack opening displacement (COD), has been employed almost exclusively to the problem of scattering from *closed* or *tight* cracks (see, e.g., Refs. 7, 12, and 13). To the authors' best knowledge, no BIE solutions have been reported in the literature for the problem of *elastic* wave scattering from *open* cracks or *thin* voids, or *thin* inclusions. On the other hand, many real problems and experimental calibrations deal with open cracks, notches, and rough cracks with asperities, for which the ideal, *one surface model* is insufficient. Further, many situations of interest involve scattering from thin inclusions and thin-layer interfaces where a shell-like one surface model or lumped-parameter model of the thin region is inadequate. The present study aims to fill this gap and provide a BIE modeling tool for thin shapes with more realistic geometry, by using the composite BIE formulation.

This composite BIE formulation applied in this study was originally developed in Ref. 20 to overcome the fictitious eigenfrequency difficulty existing in the conventional BIE formulation of the *exterior elastic wave* problems. To avoid the hypersingular integrals, the hypersingular BIE is recast in a form in which all integrals are at most weakly singular and thus no special numerical schemes are needed. Nearly singular and nearly hypersingular integrals in this composite BIE formulation, which arise when parts of the boundary surface become close to one another, as is the case for thin shapes, are transformed into sums of weakly singular integrals and nonsingular line integrals. Thus, no finer mesh is needed to deal with these nearly singular integrals. In order to demonstrate the effectiveness of the composite BIE formulation for problems with thin shapes, numerical examples of scattering from penny-shaped cracks with varying openings are given for both longitudinal and transverse incident waves. Results from these numerical example show that the composite BIE formulation is very stable for all ranges of the thinness of a thin shape, even when the two surfaces touch each other. It is also shown, as already demonstrated in acoustics,[2,3,14] that scattered fields for an open crack with an opening of $2h$ will depart noticeably from those fields for a tight crack when h is larger than 5% of the radius of the crack, especially for plane shear waves with an oblique incident angle. All these suggest that the composite BIE formulation, as proposed in this paper for analyzing thin shapes, is not only robust but also useful in providing information about the physics of such problems.

I. THE COMPOSITE BIE FORMULATION

Consider a region (a body or a void) with the boundary S and immersed in a 3-D infinite, linear elastic medium. The conventional BIE (CBIE) for problems of scattering of time-harmonic waves in the *exterior* domain can be written in the following weakly singular form[18] (index notation is used in this paper):

$$u_i(P_0)+\int_S[T_{ij}(P,P_0)-\overline{T_{ij}}(P,P_0)]u_j(P)dS(P)$$
$$+\int_S\overline{T_{ij}}(P,P_0)[u_j(P)-u_j(P_0)]dS(P)$$
$$=\int_S U_{ij}(P,P_0)t_j(P)dS(P)+u_i^I(P_0),\quad \forall P_0\in S, \quad (1)$$

where u_i and t_i are the total displacement and traction vectors, respectively, U_{ij} and T_{ij} the two dynamic kernels (dependence on the frequency is implied), $\overline{T_{ij}}$ the static kernel, u_i^I the displacement vector of the incident wave, P and P_0 the field and source points, respectively. For interior problems, the free terms $u_i(P_0)$ and $u_i^I(P_0)$ in Eq. (1) will not be present.

The hypersingular BIE (HBIE), or traction BIE, can be written in the following weakly singular form,[20]

$$t_i(P_0)+\int_S\overline{H_{ij}}(P,P_0)$$
$$\times\left[u_j(P)-u_j(P_0)-\frac{\partial u_j}{\partial\xi_\alpha}(P_0)(\xi_\alpha-\xi_{0\alpha})\right]dS(P)$$
$$+\int_S[H_{ij}(P,P_0)-\overline{H_{ij}}(P,P_0)]u_j(P)dS(P)$$
$$+E_{jkpq}e_{\alpha q}\frac{\partial u_p}{\partial\xi_\alpha}(P_0)\int_S[\overline{K_{ij}}(P,P_0)n_k(P)$$
$$+\overline{T_{ji}}(P,P_0)n_k(P_0)]dS(P)$$
$$=\int_S[K_{ij}(P,P_0)+\overline{T_{ji}}(P,P_0)]t_j(P)dS(P)$$
$$-\int_S\overline{T_{ji}}(P,P_0)[t_j(P)-t_j(P_0)]dS(P)+t_i^I(P_0),$$
$$\forall P_0\in S, \quad (2)$$

in which H_{ij} and $\overline{H_{ij}}$ are the dynamic and static hypersingular kernels, respectively, K_{ij} and $\overline{K_{ij}}$ another pair of singular kernels, n_k the components of the normal, E_{ijkl} the elastic modulus tensor, ξ_α and $\xi_{0\alpha}$ ($\alpha=1,2$) the two (tangential) coordinates of the points P and P_0, respectively, in a local curvilinear coordinate system defined on the surface S and $e_{\alpha k}=\partial\xi_\alpha/\partial x_k$ evaluated at $\xi_\alpha=\xi_{0\alpha}$ ($k=1,2,3$). Details of the derivation, notation and expressions of all the kernel functions in Eq. (2) can be found in Ref. 20. This HBIE was originally proposed to deal with problems of scattering and radiation from *bulky-shaped* objects.

Both the CBIE and HBIE will degenerate, i.e., become unsolvable or ill conditioned, when they are applied *individually* to thin shapes, e.g., to crack-like problems (imagine that a bulky void becomes a thin void or open crack) as well as true-crack problems,[2,3] see Fig. 1. This degeneracy is manifested by the fact that algebraic equations generated from the BIE on one surface of the crack are (almost) the same as the equations generated on the other surface of the crack. The condition number of the system of equations will increase sharply as the two surfaces of the crack become close. As discussed in Refs. 2 and 3, one remedy to this degeneracy associated with the crack-like (or thin-body) problem is to apply CBIE on one surface of the crack and HBIE on the other surface. This approach, using two surfaces in the model, has been demonstrated to be very effective for *acoustic* wave scattering from thin rigid screens[2] and thin inclusions.[3] Alternatively, and perhaps more advantageous due to the symmetry, it was found that Burton and Miller's composite BIE formulation,[15] using a linear combination of the CBIE and the HBIE as shown symbolically by

$$\text{CBIE}+\beta\text{HBIE}=0 \quad (3)$$

(β is the coupling parameter) will not degenerate when it is applied on both surfaces of a thin void or thin body,[3] see Fig. 2. This composite formulation was originally proposed in Ref. 15 to overcome the fictitious eigenfrequency difficulty (FED) existing in the BIE formulations for exterior *acoustic* wave problems. Recent implementations of this composite BIE formulation to deal with the FED, with weakly singular forms of the HBIEs as key ingredients, can be found in Ref. 20 for *elastic* wave problems and in Ref. 21 for *acoustic* wave problems.

The composite BIE formulation (3), in the context of *elastic* wave problems, using the linear combination of Eq. (1) (CBIE) and Eq. (2) (HBIE), is employed in this study to investigate the problem of elastic wave scattering from thin shapes (e.g., open cracks) in 3 D. This composite BIE formulation is quite general and can be applied to many other thin body problems, such as thin inclusions, thin-layer interfaces in composites and so on. The main objective here is to demonstrate the advantages of this composite formulation which can overcome both the fictitious eigenfrequency difficulty and the thin body difficulty in the conventional BIE formulation. The acoustic wave counterpart of this formulation (Burton and Miller's BIE formulation) has the same feature. Thus this composite BIE formulation can provide unique solutions for scattering from bodies with any shapes (including thin shapes) and at any frequencies. This means that one does not need to switch BIE formulations when dealing with fictitious eigenfrequency and thin body problems.

II. NEARLY SINGULAR AND HYPERSINGULAR INTEGRALS

To apply the composite BIE formulation to thin-body problems, one has to overcome another difficulty, i.e., the *nearly singular* and *nearly hypersingular integrals* which

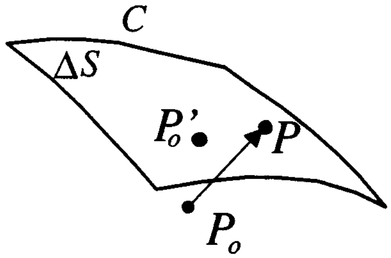

FIG. 3. Nearly singular integrals on surface ΔS enclosed by line C.

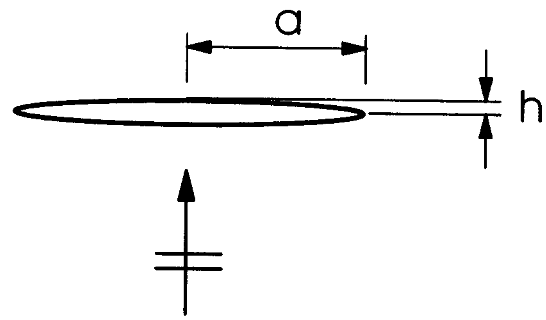

FIG. 4. A penny-shaped open crack with normal incident longitudinal wave.

arise when collocation point is on one surface of the thin body and integration need to be performed on the other nearby surface. Many numerical schemes for computing the nearly singular integrals can be found in the literature of the boundary element method (BEM). Among all the available methods, the line integral approach, i.e., transforming the nearly singular integral into a sum of weakly singular integrals and nonsingular line integrals, is believed to be the most effective and efficient one.[22] It is the method adopted and implemented in this study.

A typical nearly singular integral in Eq. (1) (CBIE) is the one with the stress kernel function T_{ij} and integrated on a surface ΔS with source point P_0 nearby, Fig. 3. Here ΔS can be one element or several elements on the surface S. This nearly singular integral can be dealt with by adding and subtracting terms in the following manner:

$$\begin{aligned}\int_{\Delta S} T_{ij}(P,P_0)u_j(P)dS(P) &= \int_{\Delta S}[T_{ij}(P,P_0)-\overline{T_{ij}}(P,P_0)]u_j(P)dS(P) \\ &\quad + \int_{\Delta S}\overline{T_{ij}}(P,P_0)u_j(P)dS(P) \\ &= \int_{\Delta S}[T_{ij}(P,P_0)-\overline{T_{ij}}(P,P_0)]u_j(P)dS(P) \\ &\quad + \int_{\Delta S}\overline{T_{ij}}(P,P_0)[u_j(P)-u_j(P_0')]dS(P) \\ &\quad + u_j(P_0')\int_{\Delta S}\overline{T_{ij}}(P,P_0)dS(P), \end{aligned} \tag{4}$$

in which P_0' is the closest point on ΔS to P_0 (an image point of P_0 on ΔS), see Fig. 3. The first two integrals in (4) are now at most nearly weakly singular and can be computed using the normal quadrature. The last integral in (4) can be transformed into line integrals as follows.[22]

$$\int_{\Delta S}\overline{T_{ij}}(P,P_0)dS(P) = I_\Omega(P_0)\delta_{ij} + \frac{1}{4\pi}\epsilon_{ijk}\oint_C \frac{1}{r}\,dx_k + \frac{1}{8\pi(1-\nu)}\epsilon_{jkl}\oint_C r,_{ik}\,dx_l, \tag{5}$$

where $r=|\overrightarrow{P_0P}|$, ν is Poisson's ratio, C is the boundary curve of ΔS, ϵ_{ijk} is the permutation tensor, and

$$I_\Omega(P_0) = -\frac{1}{4\pi}\int_{\Delta S}\frac{1}{r^2}\frac{\partial r}{\partial n}\,dS$$

is a solid angle integral which can also be evaluated using a line integral. All these line integrals are nonsingular at all since the source point P_0 is always off the contour C. The nearly *hypersingular* integrals presented in Eq. (2) (HBIE) can be dealt with in a similar way. The expressions of the line integrals for integrals involving the *static* hypersingular kernels can be found in Ref. 22.

Using this line integral approach to deal with the nearly singular integrals is very efficient in computation. One does not need to use more elements in the model in order to handle these nearly singular integrals. It was found that the CPU time used to compute these nearly singular integrals using the line integral approach is only a fraction of that when using many subdivisions on the original surface elements, in achieving the same accuracy.[22]

III. NUMERICAL EXAMPLES

To demonstrate the effectiveness of the proposed composite BIE formulation for problems involving thin shapes, the problem of elastic wave scattering from *open cracks* is studied, for which an analytical solution[23] is available when the opening is zero.

In the first case, a penny-shaped open crack with radius a and thickness $2h$ in a 3-D elastic medium is impinged upon by a plane *longitudinal wave* in the normal direction, see Fig. 4. The scattering cross section for various openings at the nondimensional (shear) wave numbers $K_Ta = 1,2,\ldots,6$ are computed using the composite BIE and compared with the analytical solution[23] (only a limited number of data points are available) which is valid for true tight cracks (with zero opening).

Figure 5 shows the results for a very small opening ($h=0.000001a$) using an increasing number (M) of *nonconforming* quadratic boundary elements[7,20,21] on the two surfaces of the crack. As expected, the BIE solutions are converging to the analytical solution for the tight crack. The small difference is probably due to the fact that the singularity feature of the field near the crack tip is not built in the boundary elements in that region. The singularity feature near the tip of an *open* crack, which depends on the ''open-

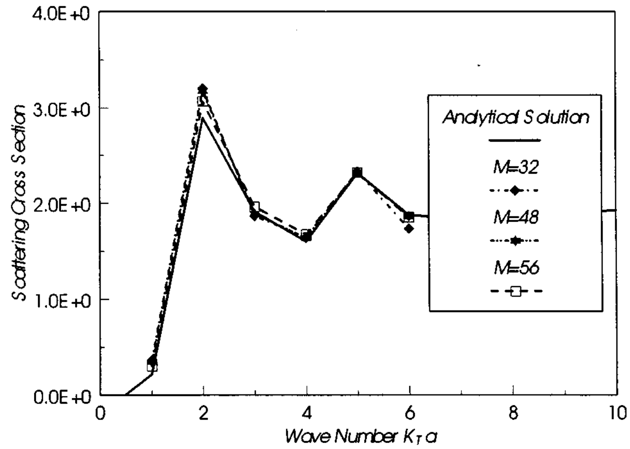

FIG. 5. Convergence of BIE solutions for $h=0.000001a$, longitudinal wave.

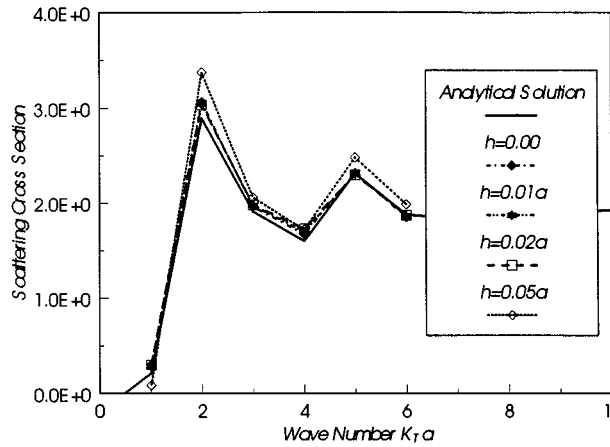

FIG. 7. BIE solutions using 56 elements at different openings, longitudinal wave.

ing angle'' at the crack tip, is difficult to implement and hence is not attempted in this study for elements near the crack tips.

Figure 6 is a similar plot, but for a larger opening ($h=0.02a$). The BIE results are still converging to the tight crack solution. This shows that the small opening does not have a noticeable effect on the far-field data, at least for a plane longitudinal wave in the normal direction of the crack.

Figure 7 shows the results for the crack with four different openings using 56 boundary elements. Noticeable departure of the BIE solution from the analytical solution for tight cracks is observed at the opening $h=0.05a$. This is the departure point for the longitudinal wave in the normal direction. When $h>0.1a$, further departures from the tight crack solution can be observed (not shown here), and one can choose either CBIE *alone* or HBIE *alone* to solve the problem since the degeneracy associated with them is not so severe.

In the next case, the open penny-shaped crack is impinged upon by plane shear waves at different angles of incidence, see Fig. 8. The scattering cross sections are computed, using the composite BIE, at each angle (ranging from 0° to 90° with a 15° increment) of incidence and compared with the analytical solution given in Ref. 23 for a tight crack.

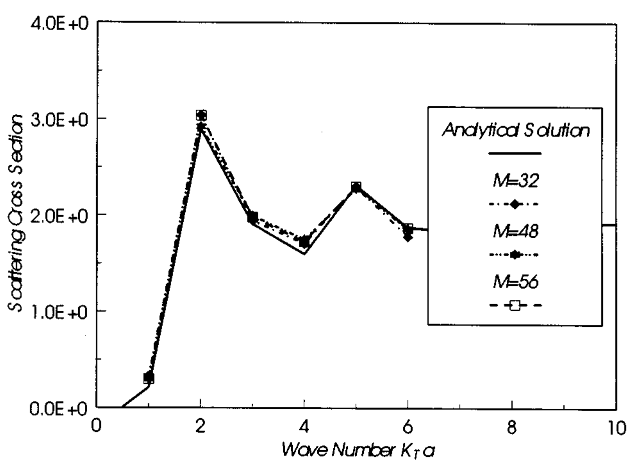

FIG. 6. Convergence of BIE solutions for $h=0.02a$, longitudinal wave.

Figure 9 shows the convergence of the composite BIE solutions for the opening $h=0.000001a$ and at the wave number $K_T a=4$. The small gap between the BIE solutions and the analytical solution maybe once again due to the singularity near the crack tip which is not implemented in the boundary elements.

Figure 10 is a plot of the BIE solutions using 56 boundary elements for different openings and at $K_T a=4$. Unlike the case of normal incidence of longitudinal wave (Fig. 7), significant departure of the BIE solution at $h=0.05a$ from the analytical solution for tight cracks is observed. Similar phenomenon is present at a lower wave number ($K_T a=3$), as shown in Fig. 11.

In all the computations, the systems of equations using the composite BIE formulation are well behaved (condition numbers are low). The choice of the coupling parameter β used in Eq. (3) is not so restrictive and values between -1 to $+1$ are found to be adequate.

IV. CONCLUSION

The composite boundary integral equation formulation, using a linear combination of CBIE and HBIE, is proposed for elastic wave problems involving thin shapes (open cracks or thin voids, thin inclusions, thin layer interfaces, etc.) modeled with *two surfaces*. The BIE formulation is very stable no matter how close the two surfaces are, and no undue

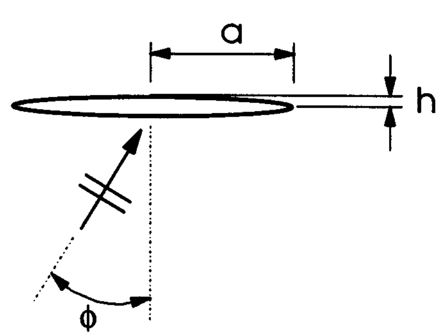

FIG. 8. A penny-shaped open crack with oblique incident shear waves.

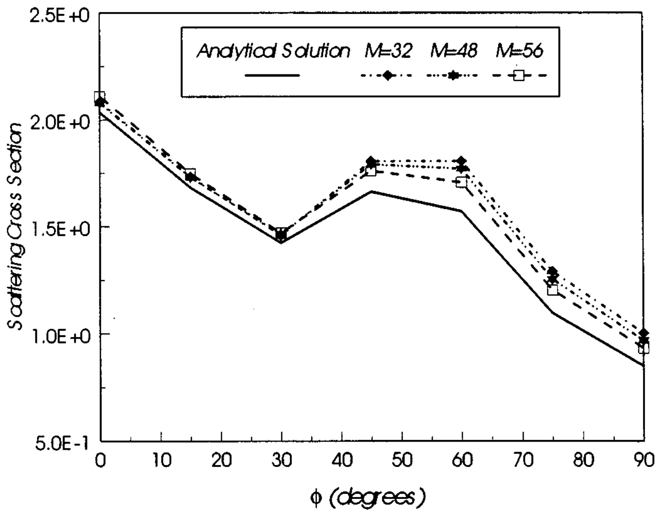

FIG. 9. Convergence of BIE solutions for $h=0.000001a$, shear wave ($K_T a=4$).

numerical burden is associated with the nearly singular integrals in this approach. This composite BIE is demonstrated to be very effective in the study of scattering from open cracks. Preliminary numerical results show that scattered fields for an open crack with an opening of $2h$ will depart significantly from those fields for a tight crack when h is larger than 5% of the radius of the crack, especially for plane shear waves. All these suggest that the composite BIE formulation is not only robust but also useful in providing information about the physics of thin shape problems. It can fill the gap between the current available one surface models for thin shapes and the real situations or experimental calibrations dealing with open cracks, notches, and rough cracks. The composite BIE formulation is especially valuable for problems in which both the fictitious eigenfrequency difficulty and the thin body breakdown difficulty have to be dealt with.

The composite BIE formulation developed in this paper can be applied to studies of scattering from thin inclusions in materials, thin layer interfaces or interface open (as well as

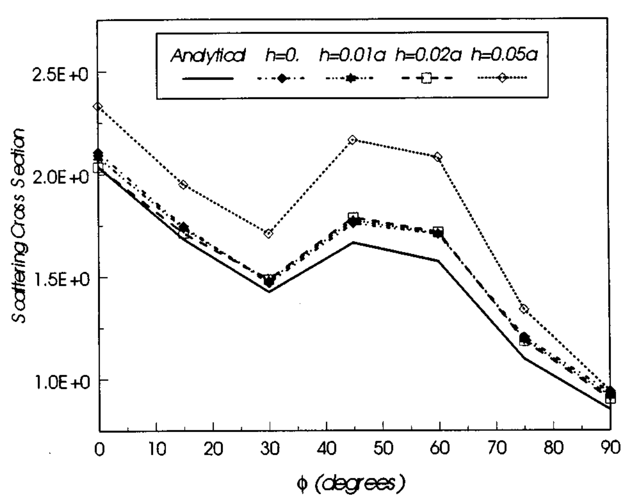

FIG. 10. Solutions using 56 elements at different openings, shear wave ($K_T a=4$).

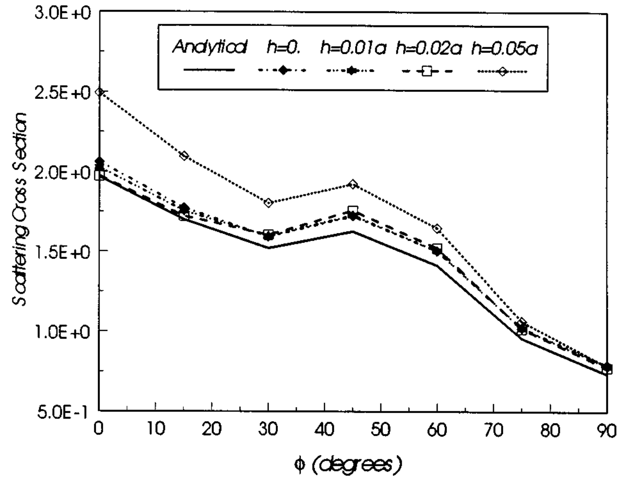

FIG. 11. Solutions using 56 elements at different openings, shear wave ($K_T a=3$).

closed) cracks, fluid-thin shell like structure interactions, all of which are demanding problems. More complicated numerical example problems are being studied along these lines and the results will be reported in future papers.

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