

Hypersingular boundary integral equations for radiation and scattering of elastic waves in three dimensions

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A weakly singular form of the hypersingular boundary integral equation (BIE) (traction equation) for 3-D elastic wave problems is developed in this paper. All integrals involved are at most weakly singular and except for a stronger smoothness requirement on boundary elements, regular quadrature and collocation procedures used for conventional BIEs are sufficient for the discretization of the original hypersingular BIE. This weakly singular form of the hypersingular BIE is applied to the composite BIE formulation which uses a linear combination of the conventional BIE and the hypersingular BIE to remove the fictitious eigenfrequencies existing in the conventional BIE formulation for elastic wave problems. Numerical examples employing different types of boundary elements clearly demonstrate the effectiveness and efficiency of the developed formulation.

1. Introduction

In a recent paper [1], a weakly singular form of the hypersingular BIE for 3-D *acoustic* problems was developed and applied successfully in Burton and Miller's composite BIE formulation [2] using a linear combination of the conventional BIE and the hypersingular BIE to furnish unique solutions at all frequencies. This work is extended to the case of 3-D *elastic* wave problems in the present paper.

First, a weakly singular form of the hypersingular BIE (associated with the traction) for 3-D elastic wave problems is derived. The procedure in deriving and the final result of this weakly singular form are almost parallel to those for the acoustic case. One difference is that the tangential derivatives of the density function, instead of the total gradients, are used in the two term subtraction of the density function associated with the hypersingular kernel. This 'tangential form' of the hypersingular BIE was first presented in [3] for elastostatic problems. The integral identities for the static Kelvin solution (fundamental solution) [4] are employed in the derivation of this weakly singular form.

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This weakly singular form of the hypersingular BIE for radiation and scattering problems in an elastic medium has the same advantages as those mentioned in [1] for the acoustic case. Since all integrals involved are at most weakly singular, no special integration quadratures are needed. With due attention to increased smoothness requirements, quadratures used for the conventional BIE can be applied directly. Hence the discretization of this weakly singular form of the hypersingular BIE is quite straightforward.

Second, the weakly singular form of the hypersingular BIE developed is applied in the composite BIE formulation which is a generalization of Burton and Miller's formulation originated for acoustic problems to elastic wave problems. In the literature, contrary to the acoustic case, only a few references can be found for the work of overcoming the fictitious eigenfrequency difficulty (FED) in the conventional BIE formulation of elastic wave problems. The modified kernel method was given in [5] without numerical implementation. The BIFILM method [6, 7], which is a variation of the CHIEF method [8] for acoustic problems, was developed and used successfully to some extent in [7, 9]. Most significantly, the composite BIE formulation for elastic wave problems was developed and the uniqueness of solution at all wave numbers, regardless of the type of boundary conditions or type of input waves, was proved by Jones [10]. A transformation was used in [10] to reduce the order of singularity of the hypersingular BIE but that transformation is limited to the use of flat patches (constant elements); also, no numerical results were reported.

In the present paper, numerical examples are provided using the composite BIE formulation, with the weakly singular form of the hypersingular BIE as key ingredient. No limitation to the use of flat elements is present here. Two types of boundary elements are implemented and tested for this composite BIE formulation, namely, non-conforming quadratic elements and the Overhauser C^1 continuous elements which were developed in [11, 12] and implemented for acoustic problems in [1, 13]. The numerical results clearly show the effectiveness and efficiency of the developed composite BIE formulation in overcoming the fictitious eigenfrequency difficulty for elastic wave problems.

2. Weakly singular form of the hypersingular BIE

Consider elastic waves in linear, homogeneous and isotropic media. The starting point is the following representation integral involving boundary values of time-harmonic elastodynamic variables [6] (index notation is used in this paper)

$$C_{ij}(P_o)u_j(P_o) = \int_S [U_{ij}(P, P_o)t_j(P) - T_{ij}(P, P_o)u_j(P)] dS(P) + u_i^1(P_o), \quad (1)$$

where u_i and t_i are the total displacement and traction components, respectively, u_i^1 is the incident displacement field (for a scattering problem), $U_{ij}(P, P_o)$ and $T_{ij}(P, P_o)$ are the displacement and traction tensors, respectively, of the time-harmonic Kelvin solution which is frequency dependent, the coefficient tensor $C_{ij}(P_o) = \delta_{ij}$, $\frac{1}{2}\delta_{ij}$ or 0 when the source point P_o is in the exterior domain E , on the boundary S (assume it is smooth) or in the interior domain B , respectively (Fig. 1). Equation (1) with $P_o \in S$ is the conventional BIE for elastic wave

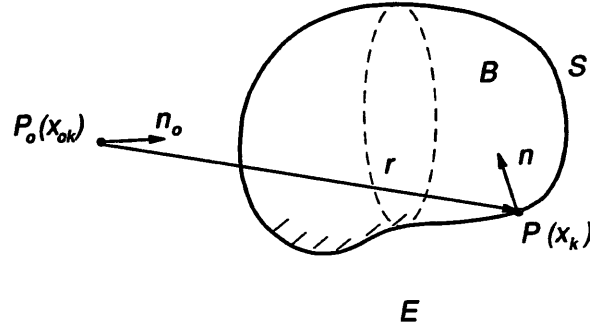


Fig. 1. Notation.

problems. This BIE contains strongly singular integrals of the CPV type and can be reformulated in a weakly singular form if an integral expression for $C_{ij}(P_o)$ is employed [4, 6].

To derive the weakly singular form of the hypersingular BIE, we take the derivatives of (1) when $P_o \in E$ and multiply both sides with the elastic modulus tensor E_{ijkl} and n_{ok} , which is the direction cosine of a vector n_o associated with P_o , to obtain the following integral expression:

$$\sigma_{ik}(P_o)n_{ok} = \int_S [K_{ij}(P, P_o)t_j(P) - H_{ij}(P, P_o)u_j(P)] dS(P) + \sigma_{ik}^1(P_o)n_{ok} \quad \forall P_o \in E, \quad (2)$$

where σ_{ik} is the stress field and

$$H_{ij}(P, P_o) = E_{ikpq} \frac{\partial T_{pj}(P, P_o)}{\partial x_{oq}} n_{ok}, \quad K_{ij}(P, P_o) = E_{ikpq} \frac{\partial U_{pj}(P, P_o)}{\partial x_{oq}} n_{ok}.$$

The second integral in (2) is hypersingular when P_o is placed on the boundary S . To regularize this hypersingular integral, we write identically for $P_o \in E$,

$$\begin{aligned} & \int_S H_{ij}(P, P_o)u_j(P) dS(P) \\ &= \int_S [H_{ij}(P, P_o) - \bar{H}_{ij}(P, P_o)]u_j(P) dS(P) + \int_S \bar{H}_{ij}(P, P_o)u_j(P) dS(P) \\ &= \int_S [H_{ij}(P, P_o) - \bar{H}_{ij}(P, P_o)]u_j(P) dS(P) \\ & \quad + \int_S \bar{H}_{ij}(P, P_o) \left[u_j(P) - u_j(P_o) - \frac{\partial u_j}{\partial \xi_\alpha}(P_o)(\xi_\alpha - \xi_{o\alpha}) \right] dS(P) \\ & \quad + u_j(P_o) \int_S \bar{H}_{ij}(P, P_o) dS(P) \\ & \quad + \frac{\partial u_j}{\partial \xi_\alpha}(P_o) \int_S (\xi_\alpha - \xi_{o\alpha}) \bar{H}_{ij}(P, P_o) dS(P) \quad \forall P_o \in E, \end{aligned} \quad (3)$$

in which

$$\bar{H}_{ij}(P, P_o) = E_{ikpq} \frac{\partial \bar{T}_{pj}(P, P_o)}{\partial x_{oq}} n_{ok},$$

where $\bar{T}_{pj}(P, P_o)$ is the traction tensor of the static Kelvin solution, ξ_α and $\xi_{o\alpha}$ (summation over α implied in (3), $\alpha = 1, 2$) are the (first two) coordinates of the field point P and the source point P_o , respectively, in a local curvilinear coordinate system $O\xi_1\xi_2\xi_3$ defined on S with ξ_1, ξ_2 in the tangential directions and ξ_3 in the normal direction (Fig. 2).

From the transformation relations, we have

$$\frac{\partial \phi}{\partial \xi_\alpha} (\xi_\alpha - \xi_{o\alpha}) = \frac{\partial \phi}{\partial x_l} e_{\alpha l} (x_l - x_{ol})$$

for a function ϕ , where $e_{\alpha l} = \partial \xi_\alpha / \partial x_l$ ($\alpha = 1, 2; l = 1, 2, 3$) are the first two column vectors of the inverse of the Jacobian. Using this fact and the following three integral identities developed in [4]:

$$\int_S \bar{T}_{pj}(P, P_o) dS(P) = 0 \quad \forall P_o \in E, \quad (4)$$

$$\int_S \frac{\partial}{\partial x_{oq}} \bar{T}_{pj}(P, P_o) dS(P) = 0 \quad \forall P_o \in E, \quad (5)$$

$$\int_S (x_l - x_{ol}) \frac{\partial}{\partial x_{oq}} \bar{T}_{pj}(P, P_o) dS(P) = E_{jlst} \int_S n_s(P) \frac{\partial}{\partial x_{oq}} \bar{U}_{pt}(P, P_o) dS(P) \quad \forall P_o \in E, \quad (6)$$

where $\bar{U}_{ij}(P, P_o)$ is the displacement tensor of the static Kelvin solution, we can evaluate the last two terms in (3) as follows:

$$u_j(P_o) \int_S \bar{H}_{ij}(P, P_o) dS(P) = 0, \quad (7)$$

and

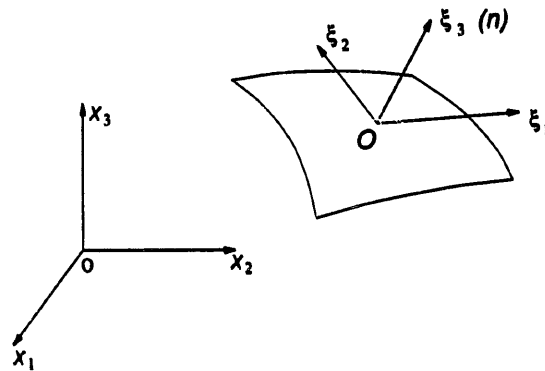


Fig. 2. Coordinate transformations.

$$\begin{aligned}
& \frac{\partial u_j}{\partial \xi_\alpha}(P_o) \int_S (\xi_\alpha - \xi_{o\alpha}) \bar{H}_{ij}(P, P_o) dS(P) \\
&= \frac{\partial u_j}{\partial \xi_\alpha}(P_o) e_{\alpha l} E_{ikpq} n_{ok} \int_S (x_l - x_{ol}) \frac{\partial}{\partial x_{oq}} \bar{T}_{pj}(P, P_o) dS(P) \\
&= \frac{\partial u_j}{\partial \xi_\alpha}(P_o) e_{\alpha l} E_{ikpq} n_{ok} E_{jlst} \int_S n_s(P) \frac{\partial}{\partial x_{oq}} \bar{U}_{pj}(P, P_o) dS(P) \\
&= \frac{\partial u_j}{\partial \xi_\alpha}(P_o) e_{\alpha l} E_{jlst} \int_S \bar{K}_{il}(P, P_o) n_s(P) dS(P) \\
&= \frac{\partial u_p}{\partial \xi_\alpha}(P_o) e_{\alpha q} E_{jkpq} \int_S \bar{K}_{ij}(P, P_o) n_k(P) dS(P) \\
&= \frac{\partial u_p}{\partial \xi_\alpha}(P_o) e_{\alpha q} E_{jkpq} \int_S [\bar{K}_{ij}(P, P_o) n_k(P) + \bar{T}_{ji}(P, P_o) n_{ok}] dS(P), \tag{8}
\end{aligned}$$

in which

$$\bar{K}_{ij}(P, P_o) = E_{ikpq} \frac{\partial \bar{U}_{pj}(P, P_o)}{\partial x_{oq}} n_{ok}.$$

The integral $\int_S \bar{T}_{ji}(P, P_o) n_{ok} dS(P)$ ($=0 \forall P_o \in E$) is added to the expression (8) so that the whole integrand will be weakly singular when P_o is placed on S since

$$\begin{aligned}
& \int_S [\bar{K}_{ij}(P, P_o) n_k(P) + \bar{T}_{ji}(P, P_o) n_{ok}] dS(P) \\
&= \int_S E_{ilpq} \frac{\partial}{\partial x_{oq}} \bar{U}_{pj}(P, P_o) [n_{ol} n_k(P) - n_l(P) n_{ok}] dS(P),
\end{aligned}$$

where $\bar{T}_{ij} = E_{jkpq} [\partial \bar{U}_{ip} / \partial x_q] n_k$, $\bar{U}_{ij} = \bar{U}_{ji}$ and $\partial \bar{U}_{ij} / \partial x_{ok} = -\partial \bar{U}_{ij} / \partial x_k$ have been applied. Substitutions of (7) and (8) into (3) give

$$\begin{aligned}
& \int_S H_{ij}(P, P_o) u_j(P) dS(P) \\
&= \int_S [H_{ij}(P, P_o) - \bar{H}_{ij}(P, P_o)] u_j(P) dS(P) \\
&+ \int_S \bar{H}_{ij}(P, P_o) \left[u_j(P) - u_j(P_o) - \frac{\partial u_j}{\partial \xi_\alpha}(P_o) (\xi_\alpha - \xi_{o\alpha}) \right] dS(P) \\
&+ E_{jkpq} e_{\alpha q} \frac{\partial u_p}{\partial \xi_\alpha}(P_o) \int_S [\bar{K}_{ij}(P, P_o) n_k(P) + \bar{T}_{ji}(P, P_o) n_{ok}] dS(P) \quad \forall P_o \in E. \tag{9}
\end{aligned}$$

Similarly, the strongly singular integral in (2) can be regularized as

$$\int_S K_{ij}(P, P_o) t_j(P) dS(P) = \int_S [K_{ij}(P, P_o) + \bar{T}_{ji}(P, P_o)] t_j(P) dS(P) - \int_S \bar{T}_{ji}(P, P_o) [t_j(P) - t_j(P_o)] dS(P) \quad \forall P_o \in E, \quad (10)$$

by the use of the first identity (4).

Substituting (9) and (10) into the integral representation (2), letting $P_o \rightarrow S$ and choosing n_o to be $n(P_o)$ (i.e. $n_{ok} = n_k(P_o)$), we obtain the weakly singular form of the hypersingular BIE, or traction BIE, as follows:

$$\begin{aligned} t_i(P_o) + \int_S \bar{H}_{ij}(P, P_o) \left[u_j(P) - u_j(P_o) - \frac{\partial u_j}{\partial \xi_\alpha}(P_o) (\xi_\alpha - \xi_{o\alpha}) \right] dS(P) \\ + \int_S [H_{ij}(P, P_o) - \bar{H}_{ij}(P, P_o)] u_j(P) dS(P) \\ + E_{j k p q} e_{\alpha q} \frac{\partial u_p}{\partial \xi_\alpha}(P_o) \int_S [\bar{K}_{ij}(P, P_o) n_k(P) + \bar{T}_{ji}(P, P_o) n_k(P_o)] dS(P) \\ = \int_S [K_{ij}(P, P_o) + \bar{T}_{ji}(P, P_o)] t_j(P) dS(P) \\ - \int_S \bar{T}_{ji}(P, P_o) [t_j(P) - t_j(P_o)] dS(P) + t_i^1(P, P_o) \quad \forall P_o \in S, \end{aligned} \quad (11)$$

where every integral is at most weakly singular. For an isotropic elastic medium, we have

$$E_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

where λ and μ are the Lamé constants. Thus, the weakly singular form of the hypersingular BIE can be written finally in the following matrix form:

$$\begin{aligned} t(P_o) + \int_S \bar{H}(P, P_o) \left[u(P) - u(P_o) - \frac{\partial u}{\partial \xi_\alpha}(P_o) (\xi_\alpha - \xi_{o\alpha}) \right] dS(P) \\ + \int_S [H(P, P_o) - \bar{H}(P, P_o)] u(P) dS(P) \\ + \int_S [\bar{K}(P, P_o) Q_\alpha(P) + \bar{T}^1(P, P_o) Q_\alpha(P_o)] dS(P) \frac{\partial u}{\partial \xi_\alpha}(P_o) \\ = \int_S [K(P, P_o) + \bar{T}^1(P, P_o)] t(P) dS(P) \\ - \int_S \bar{T}^1(P, P_o) [t(P) - t(P_o)] dS(P) + t^1(P, P_o) \quad \forall P_o \in S, \end{aligned} \quad (12)$$

in which summations over α are implied for the first and third integrals ($\alpha = 1, 2$), u and t are the displacement and traction vectors, respectively, t^1 is the traction vector corresponding to

the incident wave, and the matrices

$$Q_\alpha(P) = \lambda[e_\alpha n^i(P)]^t + \mu[e_\alpha n^i(P)] + \mu[e_\alpha^t n(P)]I, \quad \alpha = 1, 2,$$

with $e_\alpha = [e_{\alpha 1}, e_{\alpha 2}, e_{\alpha 3}]^t$ and $n(P) = [n_1(P), n_2(P), n_3(P)]^t$. The matrices $\bar{H}(P, P_o)$, $H(P, P_o)$, $\bar{K}(P, P_o)$, $K(P, P_o)$ and $\bar{T}(P, P_o)$ in (12), containing the kernel functions, are provided in Appendix A. The traction equation for 3-D exterior *elastostatic* problems can be extracted from (12) by letting $H(P, P_o) = \bar{H}(P, P_o)$ and $K(P, P_o) = \bar{K}(P, P_o)$.

A C^1 continuous requirement on the density function (the displacement vector u) is imposed on (12), at least in the neighborhood of the source point P_o , as for the acoustic case, see [1, 14, 15]. This smoothness requirement for the existence of the hypersingular integral as the source point tends to the boundary [15], is also demanded for the validity of the weakly singular form of the hypersingular BIE. Thus, theoretically speaking, only C^1 continuous boundary elements (or elements of less smoothness across the edges of elements but smooth enough in the neighborhood of the nodes, such as the non-conforming quadratic elements [1, 14]), can be employed in the discretization of (12). References [1, 15] contain considerable details which are directly relevant to the present class of problems.

The discretization of (12), although a vector equation, is quite straightforward because of the use of tangential derivatives of displacement vector, instead of the total gradient. The discretization procedure described in [1] for the acoustic wave problem can be readily generalized to the elastic wave problem here. Two types of boundary elements (Overhauser C^1 elements and non-conforming quadratic elements) are implemented and tested for (12) as well as (1). Both of them satisfy the smoothness requirement.

The composite BIE formulation is formed by a linear combination of the conventional BIE (CBIE) (1) and the hypersingular BIE (HBIE) (12), which can be expressed symbolically as $CBIE + \beta HBIE = 0$ where β is a coupling constant. In acoustic wave problems, this composite BIE formulation was proved by Burton and Miller [2] to have unique solutions at all wave numbers, provided the coupling constant β is a *complex* number. This conclusion was shown by Jones [10] to be true also for elastic wave problems. Results of numerical studies on this composite BIE formulation, which are believed to be the first reported data in the literature for elastic wave problems, are presented in the next section.

3. Numerical examples

The numerical examples presented here are for time-harmonic scattering from a traction-free spherical void of radius a , which is impinged upon by a plane longitudinal incident wave (P-wave) of unit magnitude (Fig. 3). The Overhauser C^1 continuous elements and the non-conforming quadratic elements [1, 13] are applied. The magnitudes of the scattered radial displacements at $R = 5a$ are plotted versus the angle θ for several fictitious eigenfrequencies and compared with the analytical solutions [16, 17]. In all cases, Poisson's ratio $\nu = 1/3$, M is the number of boundary elements used and N is the number of corresponding nodes.

Figure 4 shows the results at the wave number $k_L a = \pi$ (P-wave) by the conventional BIE, hypersingular BIE, and composite BIE (with coupling coefficient $\beta = 0.3i$) using 80 Overhauser elements. The result by the conventional BIE deteriorates at this fictitious eigen-

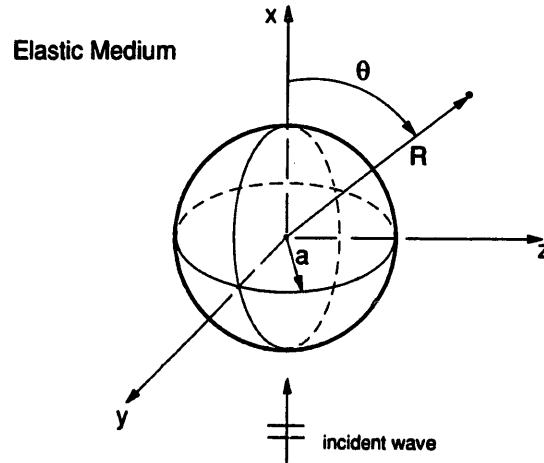


Fig. 3. The spherical void.

frequency with a large condition number of 1.5×10^4 for the system of equations, while the hypersingular and composite BIE formulations provide fairly good results with condition numbers of 86 and 27, respectively.

The results of a test on the coupling coefficient β used in the composite BIE formulation are shown in Fig. 5 for $k_L a = 4.4934$ which is also a fictitious eigenfrequency. It appears that the result is not very sensitive to the choice of the values of β , as long as β remains an imaginary number and its magnitude is not too small (we see that the results for $\beta = 1.0$ and $0.01i$ deteriorate). Although it is difficult to tell that the result is the best, the condition number is

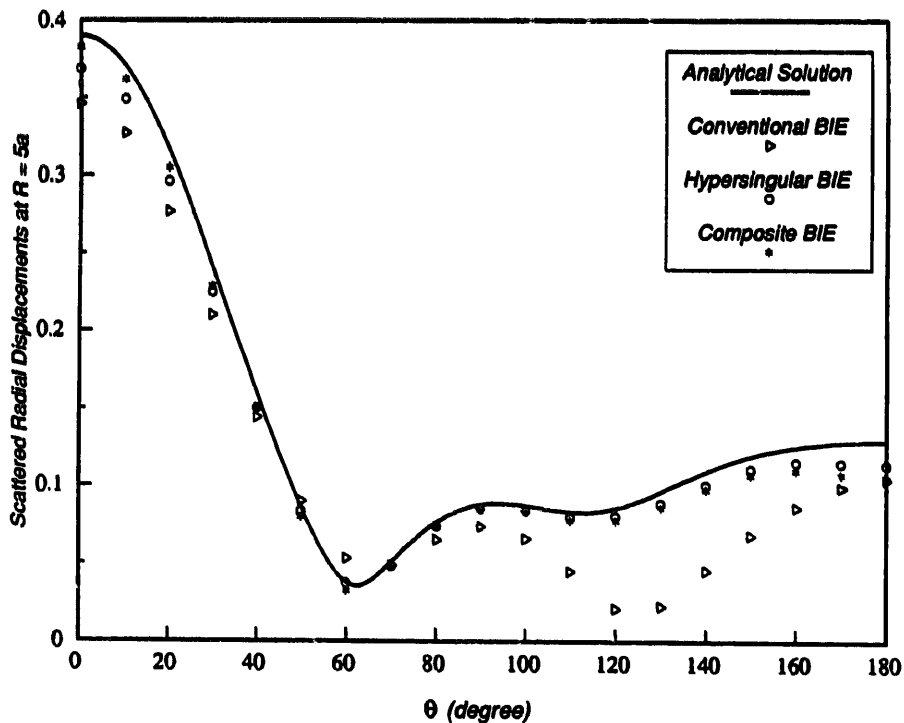


Fig. 4. The three BIE formulations, $k_L a = \pi$, Overhauser elements ($M = 80, N = 78$).

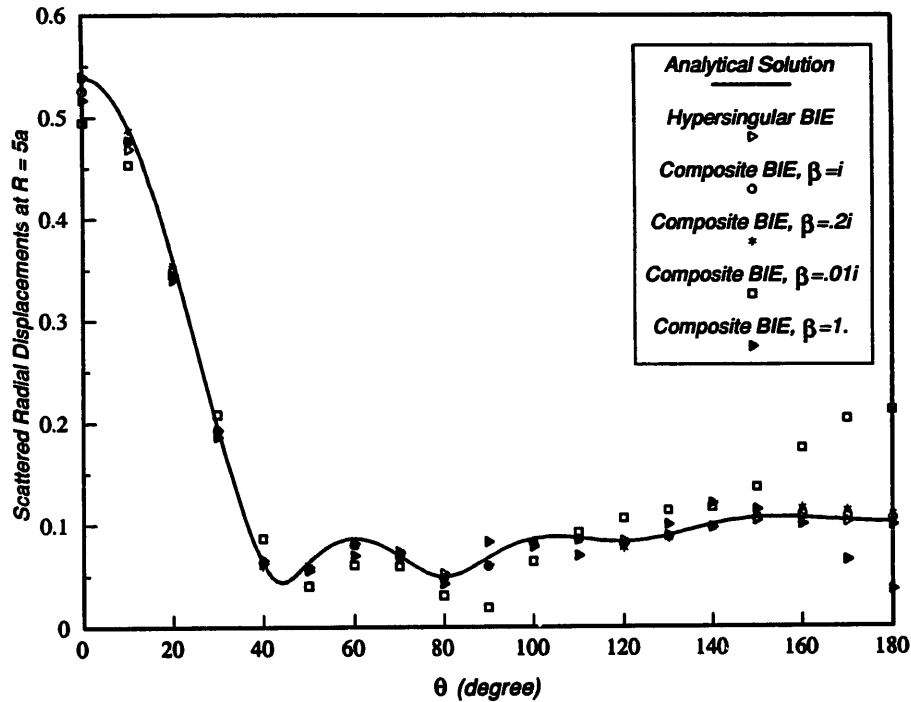


Fig. 5. Test on the coefficient β , $k_L a = 4.4934$, Overhauser elements ($M = 152$, $N = 150$).

indeed the lowest when $\beta = 0.2i$, which is in agreement with the rule of thumb, proposed for acoustic problems (see e.g. [1] for references), that β should be related to the wave number k by $\beta = i/k$ to obtain the best result. This rule of thumb has also been shown *analytically* [18, 19] to be the 'almost optimal' choice of the coupling parameter for a circle in 2-D and a sphere in 3-D in cases of acoustic and electromagnetic scattering problems. For elastic wave problems there are two wave numbers, namely, k_L for P-wave and k_T for S-wave. Wave number k_L is used here and it works well. We suspect that studies of the type [18, 19] for elastic waves would produce similar results for a near optimal parameter based on k_L .

Figure 6 is a comparison of results using the Overhauser C^1 continuous elements and the non-conforming quadratic elements, at $k_L a = 4.4934$ using the composite BIE formulation. The two meshes used here have been shown in [13] to be able to provide comparably accurate results for acoustic scattering at $ka = 2\pi$ with a ratio of about 3/4 for the total CPU time used by the Overhauser elements and the non-conforming elements. The same conclusion can be drawn here for the elastic wave scattering, but at a lower frequency, i.e. $k_L a = 4.4934$.

The frequency is increased to $k_L a = 2\pi$ in Fig. 7 and the same two meshes for Overhauser and non-conforming elements as those in Fig. 6 are used first. The results start to deteriorate for these two meshes. A finer mesh is therefore used with 80 non-conforming elements (624 nodes). However, no marked improvement in the results is observed. This certainly suggests that the composite BIE formulation for elastic wave problems, which is the vector counterpart of the acoustic problems, is much more demanding. Due to the demands on computer resources and some difficulty in generating fine meshes for Overhauser elements, no further attempt at using finer meshes is made (see [1] for a similar acoustic example for ka up to 5π).

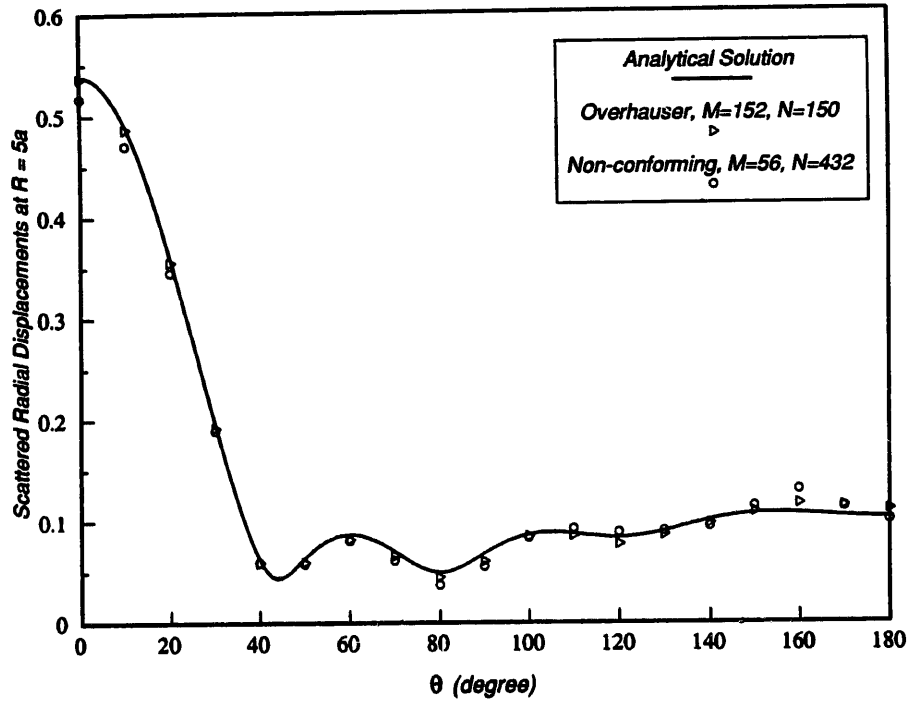


Fig. 6. Comparison of element performance, $k_L a = 4.4934$, composite BIE ($\beta = 0.2i$).

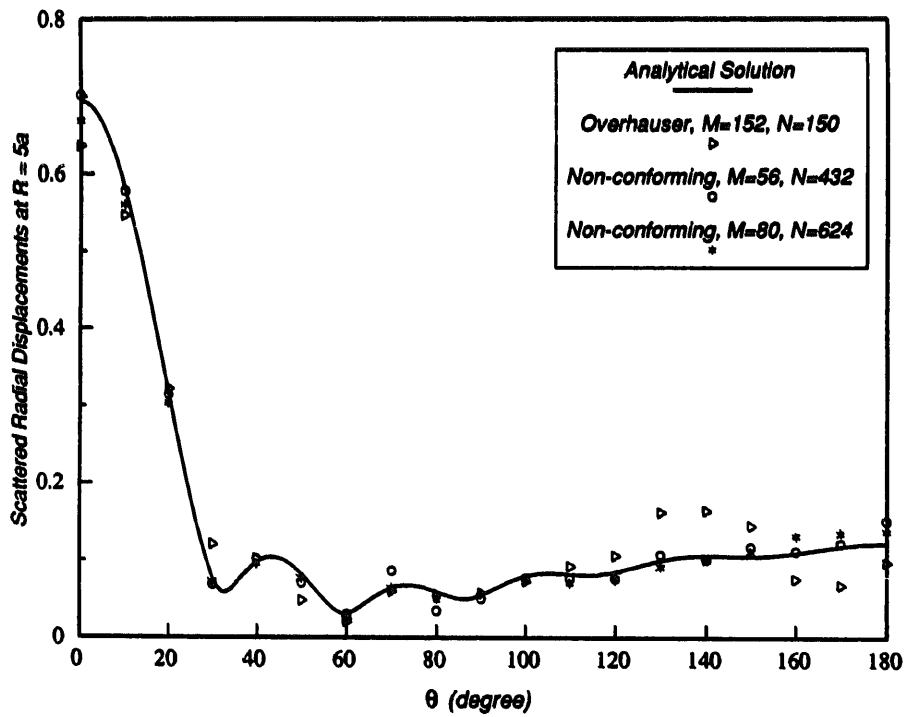


Fig. 7. Comparison of element performance, $k_L a = 2\pi$, composite BIE ($\beta = 0.15i$).

4. Discussion

The effectiveness of the composite BIE formulation, proposed by Burton and Miller [2] in the context of acoustic wave problems, and extended by Jones [10] to elastic wave problems, to overcome the fictitious eigenfrequencies, is demonstrated *numerically* for *elastic wave problems* for the first time in this paper, with the weakly singular form of the hypersingular BIE as a key ingredient.

The weakly singular form of the hypersingular BIE, presented as (12), for elastic wave problems should be used with caution with objects without smooth surfaces and continuous boundary data (e.g. traction). One safe approach is to apply (12) in these situations anyway but with non-conforming elements where all the quantities are smooth or continuous at the nodes.

The composite BIE formulations, developed in [1] for exterior acoustic wave problems and in this paper for exterior elastic wave problems, can be easily generalized to transmission problems such as fluid–fluid, fluid–solid and solid–solid problems where the fictitious eigenfrequency difficulties associated with the exterior problems also exist.

In closing, note that the hypersingular BIE, or traction BIE, is written analytically in a weakly singular form in this paper before any discretization procedure and computation are attempted. This greatly reduces the level of difficulties in applying the hypersingular BIEs and at the same time greatly increases their appeal to BIE/BEM analyses of many other engineering problems. However, it must be admitted that while difficulties with the hypersingular BIEs are reduced to manageable levels, even for vector problems, the implementation of the weakly singular form demands extra effort. Nevertheless, computing time for the formation of that part of the matrix which involves the hypersingular BIE for the present class of problems is only about 120% of that part which involves the conventional BIE. While the hypersingular BIE in the weakly singular form as described in this paper is our preference because of our experience with it to a large extent, other methods of regularization of the hypersingular BIEs are available (see the survey in [20]). Especially noteworthy among these methods, we believe, is the procedure by Guiggiani et al. [21].

Finally, we remark that the application of the hypersingular BIE to crack-like or thin-body problems and thin-inclusion problems is a very interesting and ongoing research topic [22–24]. Certainly, new areas of application of the hypersingular BIE will continue to arise as the desirable features of the hypersingular BIEs and effective ways of computing with them become more and more realized in the BIE/BEM community.

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Appendix A

The expressions for components of the kernel matrices $H(P, P_o)$, $\bar{H}(P, P_o)$, $K(P, P_o)$, $\bar{K}(P, P_o)$ and $\bar{T}(P, P_o)$ in (12) are provided in this appendix.

Let ν be Poisson's ratio, μ and λ the Lamé constants, k_L the wave number for P-waves, k_T the wave number for S-waves, and $\kappa = k_T/k_L$. Using $x = k_L r$ and $y = k_T r$, where $r = |\overline{P_o P}|$, we define the following functions:

$$C_1 = (3 - 3ix - x^2)e^{ix}, \quad D_1 = (ix - 1)e^{ix},$$

$$F_1 = (15 - 15ix - 6x^2 + ix^3)e^{ix}, \quad G_1 = (105 - 105ix - 45x^2 + 10ix^3 + x^4)e^{ix},$$

and C_2, D_2, F_2 and G_2 with y replacing x ;

$$C = C_2 - C_1, \quad D = D_2 - D_1, \quad F = F_2 - F_1, \quad G = G_2 - G_1;$$

$$\bar{C} = \frac{C}{y^2} = 3I(x, y) + \left(\frac{1}{\kappa^2} e^{ix} - e^{iy}\right), \quad \bar{D} = \frac{D}{y^2} = -I(x, y),$$

$$\bar{F} = \frac{F}{y^2} = 15I(x, y) + \left[(-6 + iy)e^{iy} - \frac{1}{\kappa^2}(-6 + ix)e^{ix}\right],$$

$$\bar{G} = \frac{G}{y^2} = 105I(x, y) + \left[(-45 + 10iy + y^2)e^{iy} - \frac{1}{\kappa^2}(-45 + 10ix + x^2)e^{ix}\right],$$

where

$$I(x, y) = \frac{1}{y^2} [(1 - iy)e^{iy} - (1 - ix)e^{ix}], \quad \text{for } y \geq y_{cr},$$

or

$$I(x, y) = 1 - \frac{1}{\kappa^2} - (1 - iy) \sum_{n=2}^{\infty} \frac{1}{n!} (iy)^{n-2} + \frac{1}{\kappa^2} (1 - ix) \sum_{n=2}^{\infty} \frac{1}{n!} (ix)^{n-2}, \quad \text{for } y \leq y_{cr},$$

where y_{cr} is a small number (e.g. 0.01).

Thus, we have the following expressions for the components $H_{ij}(P, P_o)$, $\bar{H}_{ij}(P, P_o)$, $K_{ij}(P, P_o)$, $\bar{K}_{ij}(P, P_o)$ and $\bar{T}_{ij}(P, P_o)$ of $H(P, P_o)$, $\bar{H}(P, P_o)$, $K(P, P_o)$, $\bar{K}(P, P_o)$ and $\bar{T}(P, P_o)$, respectively,

$$H_{ij}(P, P_o) = \frac{\mu}{4\pi(1-\nu)r^3} \left\{ (1-\nu) \left[(4\bar{F} - C_2)(r_{,k}n_{ok}) \frac{\partial r}{\partial n} - 2(2\bar{C} + D_2)(n_k n_{ok}) \right] \delta_{ij} \right.$$

$$- 2(1-\nu)(2\bar{C} + D_2)n_i n_{oj}$$

$$+ \left[\frac{2\nu^2}{1-2\nu} k_L^2 r^2 e^{ik_L r} - 4\nu D_1 - 4(1-\nu)\bar{C} \right] n_{oi} n_j$$

$$+ 2[2(1-\nu)\bar{F} - \nu C_1](r_{,k}n_{ok})r_{,i}n_j + (1-\nu)(4\bar{F} - C_2)(r_{,k}n_{ok})n_i r_{,j}$$

$$+ (1-\nu)(4\bar{F} - C_2) \frac{\partial r}{\partial n} r_{,i}n_{oj} + 2[2(1-\nu)\bar{F} - \nu C_1] \frac{\partial r}{\partial n} n_{oi} r_{,j}$$

$$\left. + (1-\nu) \left[(4\bar{F} - C_2)(n_k n_{ok}) - 4\bar{G}(r_{,k}n_{ok}) \frac{\partial r}{\partial n} \right] r_{,i} r_{,j} \right\},$$

$$\begin{aligned} \bar{H}_{ij}(P, P_o) &= \frac{\mu}{4\pi(1-\nu)r^3} \left\{ \left[3\nu(r_{,k}n_{ok}) \frac{\partial r}{\partial n} + (1-2\nu)(n_k n_{ok}) \right] \delta_{ij} + (1-2\nu)n_i n_{oj} \right. \\ &\quad - (1-4\nu)n_{oi}n_j + 3(1-2\nu)(r_{,k}n_{ok})r_{,i}n_j \\ &\quad + 3\nu(r_{,k}n_{ok})n_i r_{,j} + 3\nu \frac{\partial r}{\partial n} r_{,i}n_{oj} \\ &\quad \left. + 3(1-2\nu) \frac{\partial r}{\partial n} n_{oi}r_{,j} + 3 \left[\nu(n_k n_{ok}) - 5(r_{,k}n_{ok}) \frac{\partial r}{\partial n} \right] r_{,i}r_{,j} \right\}, \\ K_{ij}(P, P_o) &= \frac{1}{8\pi(1-\nu)r^2} \left\{ -2(1-\nu)(2\bar{C} + D_2)(r_{,k}n_{ok})\delta_{ij} - 2(1-\nu)(2\bar{C} + D_2)r_{,i}n_{oj} \right. \\ &\quad \left. - 2[2(1-\nu)\bar{C} + \nu D_1]n_{oi}r_{,j} + 4(1-\nu)\bar{F}(r_{,k}n_{ok})r_{,i}r_{,j} \right\}, \\ \bar{K}_{ij}(P, P_o) &= \frac{1}{8\pi(1-\nu)r^2} \left[(1-2\nu)(r_{,k}n_{ok})\delta_{ij} + (1-2\nu)r_{,i}n_{oj} - (1-2\nu)n_{oi}r_{,j} \right. \\ &\quad \left. + 3(r_{,k}n_{ok})r_{,i}r_{,j} \right] \end{aligned}$$

and

$$\begin{aligned} \bar{T}_{ij}(P, P_o) &= -\frac{1}{8\pi(1-\nu)r^2} \left[(1-2\nu) \frac{\partial r}{\partial n} \delta_{ij} - (1-2\nu)r_{,i}n_j + (1-2\nu)n_i r_{,j} \right. \\ &\quad \left. + 3 \frac{\partial r}{\partial n} r_{,i}r_{,j} \right]. \end{aligned}$$

It is noticed that as wave numbers tend to zero, i.e. as x and $y \rightarrow 0$,

$$\begin{aligned} C_1 = C_2 = 3, \quad D_1 = D_2 = -1, \\ \bar{C} = \frac{1}{4(1-\nu)}, \quad \bar{F} = \frac{3}{4(1-\nu)}, \quad \bar{G} = \frac{15}{4(1-\nu)}. \end{aligned}$$

Applying the above results, we can check that

$$\{H_{ij}, K_{ij}\} \rightarrow \{\bar{H}_{ij}, \bar{K}_{ij}\},$$

as x and $y \rightarrow 0$, as should be the case.

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