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Some advances in boundary integral methods for wave-scattering from cracks

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Summary. This paper deals with some recent and ongoing research involving scattering of time-harmonic acoustic and elastic waves from cracks and cracklike thin scatterers. The character and treatment of the singular integral equations involved in the formulation and solution of such problems are discussed and a number of numerical examples are presented.

1 Introduction

Scattering of elastic waves from arbitrarily – shaped cracks in a linear elastic solid is a topic of continuing interest in solid mechanics, and it is of fundamental importance in the nondestructive evaluation of materials by ultrasonic methods. Formulation of scattering problems in terms of boundary integrals is popular for a number of reasons; these include the ability of such integrals to express the near and far scattered fields with accuracy and stress intensity factors with ease and simplicity. The boundary element method of numerical solution of boundary integral equations, with elements confined to the crack surface, provides an attractive approach to practical problems, especially in three dimensions.

There are two main models for cracks, namely, (i) the mathematical model where the two surfaces of the crack (before loading) occupy the same place, and (ii) a thin-crack model where the two surfaces are distinct but close together. This paper discusses the degeneracy in conventional boundary integral methods for case (i) and the near-degeneracy associated with case (ii), and it suggests certain strategies to surmount these degeneracies.

A scalar counterpart of the crack problem involves scattering of acoustic waves from arbitrarily – shaped, vanishingly – thin rigid screens (cf. [1], [2]). All of the essential ideas and methods involved in the crack problem, that are to be emphasized in this paper, pertain to this scalar problem. Thus, for ease in presentation, in the next section we formulate the scalar version of the crack problem which leads to consideration of a hypersingular integral equation. Then, we discuss a regularization strategy to do computations with this equation followed by some numerical examples in Sect. 4. In Sect. 5 we do consider the vector elastodynamic problem of scattering from a crack and present some data for an elliptical crack. Finally, we consider some of the issues involved with the thin-crack model, suggest a formulation strategy, and present some data for several scalar thin-body scattering problems.

2 Formulation

Consider a time-harmonic acoustic wave incident upon a thin screen as shown in Fig. 1. We are interested in the scattered field which satisfies the Helmholtz equation

$$c^2 \nabla^2 u_s + \omega^2 u_s = 0. \tag{1}$$

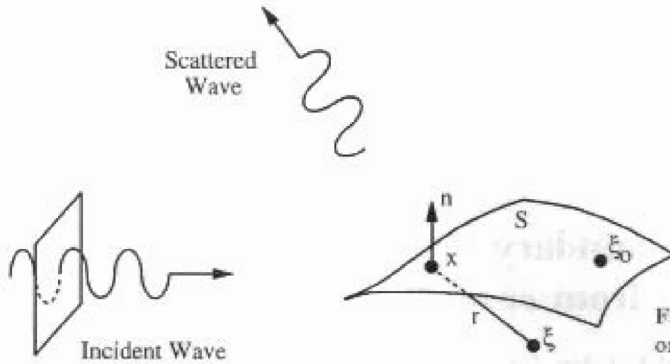

Fig. 1. Wave scattering from a screen or crack

Above, u_s is the scattered acoustic pressure which satisfies the radiation condition at infinity, c is the wave speed, ω is the frequency, and ∇ is the usual 'del' operator in three dimensional space. On the scatterer surface S, $q_s + q^I = q$ with $q_s \equiv \partial u_s / \partial n$ and $q^I \equiv \partial u^I / \partial n$, where u^I is the incident pressure and n is the normal to S (Fig. 1). Incident plus scattered field equals total field (et. seq.).

To find the scattered field, it is expedient to formulate the above problem in terms of integral equations defined on the scatterer surface. In Fig. 2 is shown a thin scatterer S_T with two surfaces S^+ and S^- identified between which there is a small but (as yet) nonzero volume. A familiar application of Green's theorem (cf. [3], [4], [5], [6], [7]) yields the identity

$$\alpha u(\xi) = \int_{S_T} \left[G(x, \xi)\, q(x) - \frac{\partial G(x, \xi)}{\partial n}\, u(x) \right] dS() + u^I(\xi) \tag{2}$$

where u and q are the total fields; ξ is a point on or off of S_T, x is the location of element of integration dS, G is the free-space Green's function $e^{ikr}/(4\pi r)$, where $k = \omega/c$ and $r = |\xi - x|$, and $\alpha = 1/2$ if ξ is on S_T (at a point ξ_0 with a well defined tangent plane) or $\alpha = 1$ if ξ is off of S but in the acoustic medium (dependence of all quantities on the frequency ω is understood). Now if the scatterer is rigid for example, such that $q = 0$, Eq. (2) with $\alpha = 1/2$ is a boundary integral equation in the unknown u on S_T. Once this equation is solved[1], formula (2) (with $\alpha = 1$), becomes the generator of u at any other ξ, as desired, to complete the solution to the scattering problem.

Next, suppose that the surfaces S^+ and S^- approach each other such that the volume between the surfaces goes to zero (forming a crack in the vector case). Then, there is difficulty with the solution process based on (2) as described above. Specifically, before the limit as S^+ goes to S^-,

Fig. 2. Thin cracklike scatterer

[1] Unique solution to (2) at certain discrete frequencies is impossible, see e.g. [8], [9], [10]. However, modifications of (2) which guarantee a unique solution at all frequencies are available. Nevertheless, for present purposes it is sufficient to note that this uniqueness difficulty disappears as the volume enclosed by S_T goes to zero.

consider the equation

$$\frac{1}{2}u^+(\xi_0^+) + \int\!\!\!\!\!\!-_{S^+} \frac{\partial G(x^+, \xi_0^+)}{\partial n^+(x^+)} u^+(x^+)\, dS(x^+) + \int_{S^-} \frac{\partial G(x^-, \xi_0^+)}{\partial n^-(x^-)} u^-(x^-)\, dS(x^-)$$

$$= \int_{S^+} G(x^+, \xi_0^+)\, q^+(x^+)\, dS(x^+) + \int_{S^-} G(x^-, \xi_0^+)\, q^-(x^-)\, dS(x^-) + u^I(\xi_0^+) \tag{3}$$

and another one just like it except with ξ_0^+ replaced ξ_0^-. Each equation represents an application of (2) (with $\alpha = 1/2$) with collocation points $\xi_0 = \xi_0^+$ and ξ_0^-, respectively. Because in the limit $G(x, \xi)$ has identical values for points on S^+ and S^-, and because in the limit $\partial G(x^+, \xi)/\partial n(x^+) = -\partial G(x^-, \xi_0)/\partial n(x^-)$ as well, both Eqs. (3) and its counterpart with $\xi_0 = \xi_0^-$ are identical, and each in the limit have the form

$$\frac{1}{2}\,\Sigma u(\xi_0) + \int\!\!\!\!\!\!-_{S} \frac{G(x, \xi_0)}{\partial n(x)} \Delta u(x)\, dS(x) = \int_S G(x, \xi_0)\, \Sigma q(x)\, dS(x) + u^I(\xi_0) \tag{4}$$

in which $\Delta u = u^+ - u^-$ is the jump in pressure across S^+ and S^-, Σq is the sum $q^+ + q^-$, and S is **either** surface S^+ or S^-. It is apparent now that if, for a rigid scatter, $q^+ = q^- = 0$, and therefore $\Sigma q = 0$, both Δu and Σu are unknown in (4) such that (4) alone is insufficient to obtain either, and the previously described solution process breaks down.

What is usually done to overcome this degeneracy is to take the normal derivative of identity (2), carefully obtain the form of this new identity corresponding to ξ_0^+ or ξ_0^- as with (3), to obtain in the limit the counterpart of (4), i.e.,

$$\frac{1}{2}\,\Delta q(\xi_0) + \int\!\!\!\!\!\!=_{S} \frac{\partial^2 G(x, \xi_0)}{\partial n(\xi_0)\,\partial n(x)} \Delta u(x)\, dS(x) = \int\!\!\!\!\!\!-_{S} \frac{G(x, \xi_0)}{\partial n(\xi_0)}\, \Sigma q(x)\, dS(x) + q^I(\xi_0). \tag{5}$$

Now setting Δq and $\Sigma q = 0$ in (5), by virtue of the (rigid) boundary condition, solve (5) for Δu such that Σu across S^+ and S^- may be obtained easily from (4). To obtain u off of the scatterer, ne.g. in the far field, the proper version of (2) (with $\alpha = 1$) to be used is

$$u(\xi) = -\int_S \frac{\partial G(x, \xi)}{\partial n} \Delta u(x)\, dS(x) + u^I(\xi). \tag{6}$$

The use of (5) with (6) and (4) works well for the infinitesimally – thin screen scatterer. References [11], [12], [13], [14], [15] and [16] are merely a sample of the recent work based on this solution strategy. An interesting feature of this strategy is the presence of the term with the 'double dash' through the integral sign in (5). The kernel of this integral involves the second derivative of the Green's function and thus is of the order $1/r^3$. By contrast, the kernel involving the first derivative of the Green's function is of order $1/r^2$, whereas the Green's function itself is of order $1/r$. These kernels are termed hypersingular, strongly or Cauchy singular, and weakly singular, respectively; and the double dash, single dash, and absence of any dash, respectively, are intended to signify the special meanings to be attached to hypersingular and strongly singular integrals (cf. [17–26]). Indeed, quite a body of literature has arisen recently in connection with hypersingular integrals which appear, as described, in the crack problem as well as other contexts (e.g. [27–29], [10]). The strongly singular and weakly singular integrals, however, are comparatively more familiar in boundary integrals analysis.

3 Regularization

There exists a variety of options for regularization (e.g. [30–33]), i.e. lowering the singularity of the integrand of hypersingular integrals before computation is attempted, or for computing them more directly (see the survey in [33]). Here we use and briefly describe only the regularization process involving a two-term Taylor series expansion, as treated more fully in [15].

If Δu is expanded in a Taylor series about ξ_0 and the first two terms are subtracted from Δu in the hypersingular integrand in (5) and added back, it is possible, with the aid of Stokes' theorem, to rewrite the hypersingular integral in (5) in the form

$$\frac{\partial u^s(\xi_0)}{\partial n(\xi_0)} = -\frac{\partial u^i(\xi_0)}{\partial n(\xi_0)} = -\int_S \left[\frac{\partial^2 G(x, \xi_0)}{\partial n(\xi_0)\, \partial n(x)} - \frac{\partial^2 G^s(x, \xi_0)}{\partial n(\xi_0)\, \partial n(x)}\right] \Delta u(x)\, dS$$

$$- \int_S \frac{\partial^2 G^s(x, \xi_0)}{\partial n(\xi_0)\, \partial n(x)} [\Delta u(x) - \Delta u(\xi_0) - \Delta u_{,p}(\xi_0)\,(x_p - \xi_{0p})]\, dS$$

$$- \Delta u_{,k}(\xi_0)\, n_r(\xi_0) \int_S \frac{\partial G^s(x, \xi_0)}{\partial \xi_r} n_k(x)\, dS$$

$$+ \Delta u(\xi_0)\, n_r(\xi_0)\, \varepsilon_{qkr} \oint_C \frac{\partial G^s(x, \xi_0)}{\partial x_k}\, dx_q$$

$$+ \Delta u_{,p}(\xi_0)\, n_r(\xi_0)\, \varepsilon_{qkr} \oint_C \frac{\partial G^s(x, \xi_0)}{\partial x_k} (x_p - \xi_{0p})\, dx_q$$

$$+ \Delta u_{,p}(\xi_0)\, n_r(\xi_0)\, \varepsilon_{qrp} \oint_C G^s(x, \xi_0)\, dx_q \tag{7}$$

wherein C is the (line) boundary of S and G^s is the static Green's function. The subtraction of G^s from G as shown is done for convenience such that the kernel in most terms is independent of frequency. No integral in (7) is more than weakly singular. Conventional boundary element methods of solving the integral equations, in use for years, may be used with proper care given to the smoothness demanded by (7) [34].

4 Numerical examples

For illustration, consider a penny-shaped rigid scatterer. The scatterer surface S^+ or S^- is discretized into 25 elements using three rings and 8 radial lines with one (circular) element in the middle. These elements used to describe the scatterer are the standard conforming quadratic elements [33], but the jump in potential, Δu, on the scatterer surface is approximated by nonconforming elements, where the collocation points are away from the element edges and have sufficient smoothness for (7) to exist. The square-root behavior of the solution along the crack edges is built into the elements at the crack edge. The Δu for inclined waves at $ka = 3$ and $ka = 4$ at 30° and 45° with the normal, respectively, is as in Figs. 3 and 4. These data are verified by

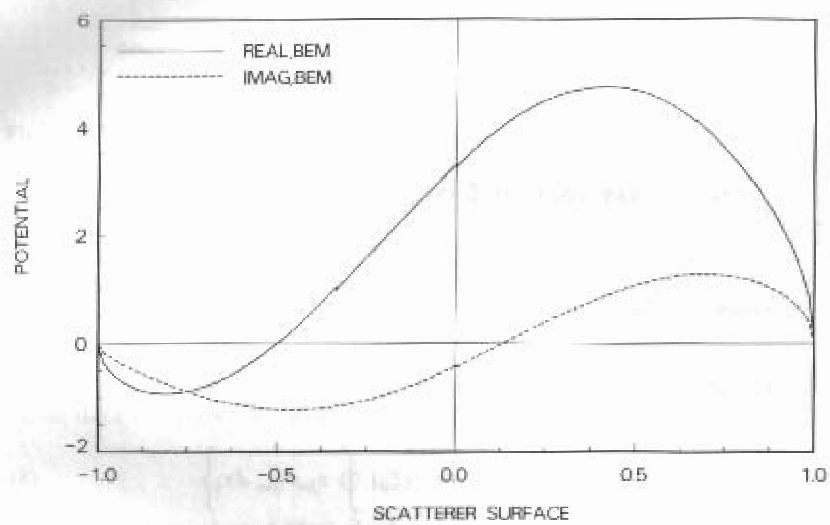

Fig. 3. Scattering due to plane wave inclined at 30° to the normal and $ka = 3.0$

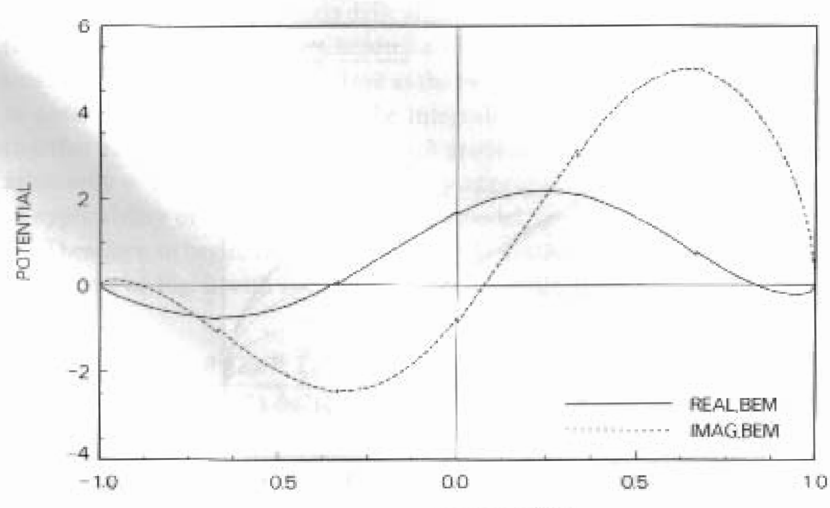

Fig. 4. Scattering due to plane wave inclined at 45° to the normal and $ka = 4.0$

solving the problems again using the essentially independent thin-body formulation as described in Sect. 6. Also, for normally incident waves, the data (see [15]) compare well with the analytical solution.

5 Vector crack problem – examples

The scattering of elastic waves from an arbitrarily shaped crack can be expressed as an integral equation,

$$
\begin{aligned}
-t_p^I(\xi_0) = C_{pqmr} n_q(\xi_0) \Bigg\{ & -C_{ijkl} \int_S \left[\frac{\partial^2 G_{km}(x, \xi)}{\partial \xi_r \, \partial x_l} - \frac{\partial^2 G^s_{km}(x, \xi)}{\partial \xi_r \, \partial x_l} \right] n_j(x) \, \Delta u_i(x) \, dS \\
& - C_{ijkl} \int_S \frac{\partial^2 G^s_{km}(x, \xi)}{\partial \xi_r \, \partial x_l} n_j(x) \left[\Delta u_i(x) - \Delta u_i(\xi_0) - \Delta u_{i,p}(\xi_0)(x_p - \xi_{0p}) \right] dS \\
& + C_{ijkl} \int_S \frac{\partial G^s_{lm}}{\partial x_r} \Delta u_{k,l}(\xi_0) \, n_j(x) \, dS + \Delta u_i(\xi_0) \, \varepsilon_{jrq} C_{ijkl} \oint_C \frac{\partial G^s_{km}}{\partial x_l} \, dx_q \\
& + \Delta u_{i,j}(\xi_0) \, \varepsilon_{rlq} C_{ijkl} \oint_C G^s_{km} \, dx_q + \Delta u_{i,p}(\xi_0) \, \varepsilon_{jrq} C_{ijkl} \oint_C \frac{\partial G^s_{km}}{\partial x_l} (x_p - \xi_{0p}) \, dx_q \\
& + \left[\frac{\Omega(\xi_0)}{4\pi} \right] \Delta u_{m,r}(\xi_0) \\
& - \frac{1}{8\pi} \Delta u_{j,r}(\xi_0) \oint_C \varepsilon_{mji} r_{,pp} \, dx_i - \frac{1}{8\pi(1-\nu)} \Delta u_{j,r}(\xi_0) \oint_C \varepsilon_{jpi} r_{,pm} \, dx_i \Bigg\}
\end{aligned}
\tag{8}
$$

where Δu_i is the crack opening displacement across the crack surface, t^I is the traction due to the incident field, G^D_{km} is the free-space, time-harmonic, elastodynamic Green's function and G^s_{km} is

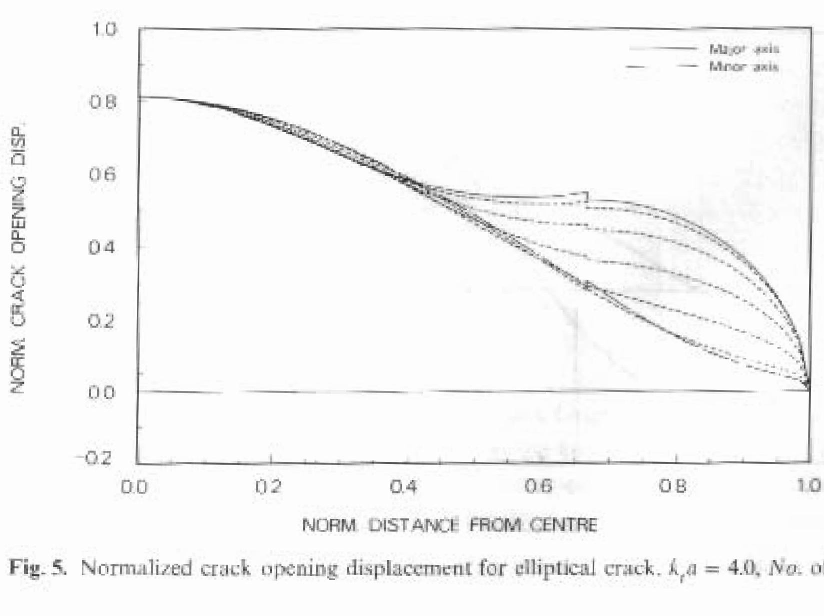

Fig. 5. Normalized crack opening displacement for elliptical crack, $k_t a = 4.0$, *No.* of elements = 25

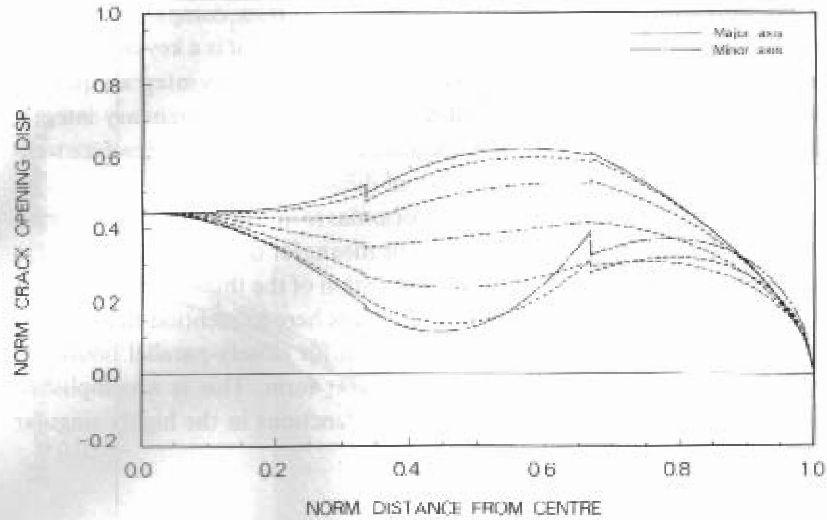

Fig. 6. Normalized crack opening displacement for elliptical crack, $k_t a = 5.0$. *No.* of elements = 25

the static equivalent of G^{D}_{km} otherwise referred to as the Kelvin solution (cf. [15]). Again C is the line which encloses the crack. For flat circular or elliptical cracks, the above equation can be solved for Δu_i. When a plane wave at $k_t a = 4.0$ and $k_t a = 5.0$ strikes an elliptical crack of aspect ratio 2 at normal incidence, Δu_m, the crack opening displacement normal to the crack surface is presented in Figs. 5 and 6. In these figures the crack opening displacement is plotted along the major and minor axis and radial lines at 15 degree intervals between the major and minor axis. These data compare well with those presented in [14].

6 Thin scatterer (cracklike) model

We return now to the scalar problem of scattering of acoustic waves, but here we examine a different scatterer, namely, a thin rigid scatterer of small but finite thickness. Our intention is to come to grips with the two main difficulties associated with the thinness of such shapes. These are (a) ill-conditioning of the formulation (i.e. (3) and its counterpart with ξ_0^+ replaced by ξ_0^-) which was shown to degenerate in the limit as the two scatterer surfaces come together, and (b) difficulty in obtaining accurate values of the integrals over boundary elements which are very close together across the small thickness. Such problems are important in a variety of cases where the arbitrarily – thin or true-crack model is inappropriate. They are useful also in testing the limits of applicability of the simpler true-crack model.

Therefore, to begin, consider the normal derivative of identity (2) again but now applied to the thin shape of Fig. 2 with specific reference to (collocation) points ξ_0^- as written below

$$\frac{1}{2} q^-(\xi_0^-) + \int_{S^+} \frac{\partial^2 G(x^+, \xi_0^-)}{\partial n^+(x^+)\,\partial n^-(\xi_0^-)} u^+(x^+)\, dS(x^+) + \oint_{S^-} \frac{\partial^2 G(x^-, \xi_0^-)}{\partial n^-(x^-)\,\partial n^-(\xi_0^-)} u^-(x^-)\, dS(x^-)$$

$$= \int_{S^+} \frac{\partial G(x^+, \xi_0^-)}{\partial n^-(\xi_0^-)} q^+(x^+)\, dS(x^+) + \int_{S^-} \frac{\partial G(x^-, \xi_0^-)}{\partial n^-(\xi_0^-)} q^-(x^-)\, dS(x^-) + q^I(\xi_0^-). \qquad (9)$$

The combination of (9) with formula (3) has very different properties compared with the combination of (3) with its counterpart referring to point ξ_0^-. Indeed, and it is a key observation for thin-body problems, that a formulation involving a conventional boundary integral equation (e.g. (3)) collocated on one surface of a thin shape and with a hypersingular boundary integral equation (e.g. (9)) collocated on the other surface is nondegenerate in the limit and therefore well-conditioned for thin shapes. This observation takes care of difficulty (a) above.

Difficulty (b) remains for thin shapes regardless of the formulas (conventional or hypersingular) used. However, it too can be surmounted, and the specific means for doing so is addressed in another paper [35] which deals in more detail with the formulation of the thin-body problem as well as more detail with numerical issues (see also [36]). It suffices here to mention that the key ingredient in developing our integration scheme, accurate even for closely-parallel boundary elements, involves reduction of all integrals to weakly singular form. This is accomplished through the two-term Taylor series expansion for the density functions in the highly singular integrands, as mentioned in Sect. 3.

7 Thin scatterer — examples

Consider the problem of scattering of acoustic waves by a thin rigid screen of thickness $2h$ as shown in Fig. 7. Waves of various frequencies and angles of incidence ϕ impinge upon the scatterer and we are interested in the scattered field for various thicknesses h. Our purpose is to examine the conditioning of our equations and the accuracy of our solution as h gets small. In the process we can compare differences between the thin body and the arbitrarily — thin model of the scatterer. To do the computations, we discretize both surfaces S^+ and S^- of the circular scatterer with elements as used for only one surface in Sect. 4.

In Fig. 8 is shown the magnitude of the scattered field for $ka = 1$, at a distance of 5 radii, as a function of θ, for a normal-incident wave, for various values of h compared with the 'arbitrarily' thin $h = 0$ model. The formulation is well conditioned at the values of h shown as well as smaller values down to $h = 10^{-7}$. Note for values of h less than about 0.1, there is little difference with the $h = 0$ model. We should mention that the $h = 0$ model has built in the square root singularity at the edges whereas the finite thickness model does not.

Therefore, for $h = 0.1$ we examined the backscatter and specular scatter as a function of incident-wave angle at various values of ka and compared with the $h = 0$ model at various distances from the crack. Specific data are not shown but in essence we found that differences with the $h = 0$ model decrease with distance and with higher frequency.

We close this section with the observation that if the thin-body shape is an inclusion, i.e. a region of fluid of (perhaps) different properties rather than a rigid inclusion, it is possible to solve this problem using a formulation with the same good properties as found above. Indeed, as a check, we modelled a thin inclusion with the same properties as the surrounding field, and obtained the homogeneous (i.e. no scatterer) solution, as expected.

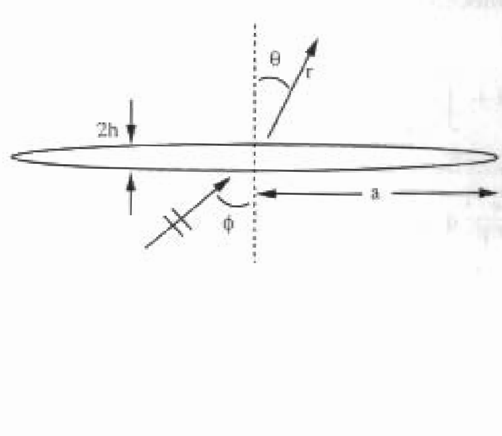

Fig. 7. Edge view of thin circular rigid scatterer

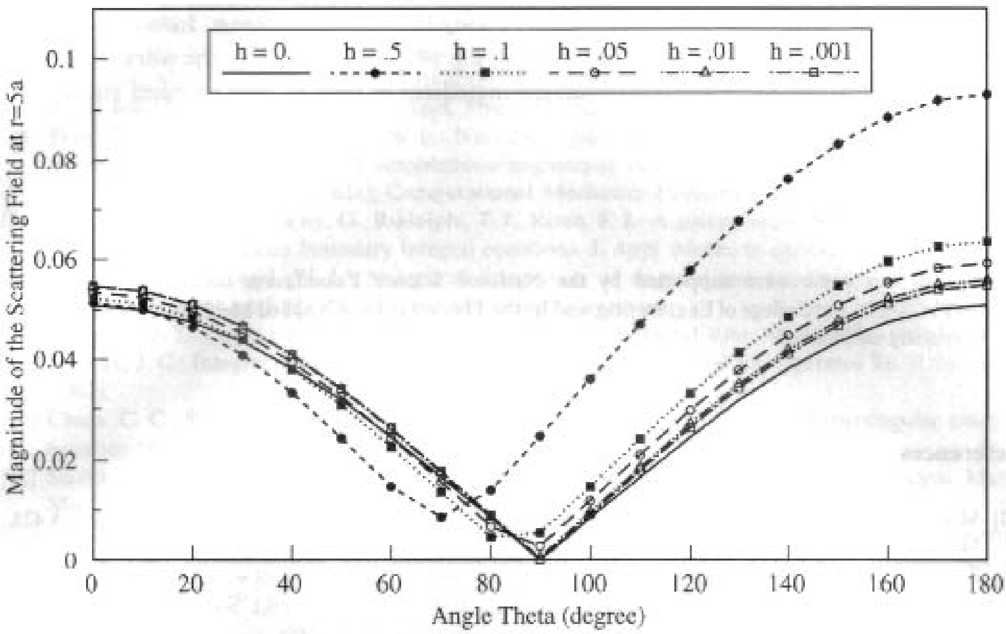

Fig. 8. Scattering for a normal incident wave, N_0 of elements = 10, N_0 of nodes = 80, $ka = 1.0$

8 Discussion

In this paper we have attempted to digest some of our recent and ongoing research with integral methods for scattering from cracks in solids and cracklike shapes in acoustic fluids. Specifically, we indicated the degeneracy in the conventional formulation for cracks or arbitrarily–thin scatterers and presented an appropriate remedy involving hypersingular integral equations. Data for some example problems for scalar and vector waves are presented following a particular strategy for dealing with (regularizing) the hypersingular integrals. Then the problems associated with scattering from thin bodies or cracklike scatterers with associated near degeneracy was addressed. Here we suggested a formulation based on collocation with conventional equations on one surface of the thin body and collocation with hypersingular equations on the other surface. Such a formulation has excellent properties and an illustrative example involving acoustic scattering from a thin penny-shaped rigid scatterer was presented for verification.

In closing we should point out some recent work [37–39] for scattering from multiple objects which involves a series approach. In this work the influence of the interactive scattering between scatterers is associated with the number of terms in a series. Thus, truncating the series neglects higher order interactive scattering. The method is shown [39] to be effective and efficient for a number of shapes, including those which are cracklike, even though the formulation involves conventional integral equations (as supposed to hypersingular) only.

All of these treatments of scattering from cracks, cracklike shapes and other shapes, in fluids or solids may be regarded as ways of solving the so called forward problem, i.e. the problem where the scatterers are known and the scattered field is unknown. The more difficult problem, and the one more technologically significant in nondestructive evaluation and target identification and characterization, is the inverse problem, wherein the object doing the scattering is unknown. It is well understood, however, that almost all strategies for the inverse problem require accurate data and efficient schemes for solving the forward problem. Indeed, boundary integral methods are already used in that capacity e.g. [40], [41]. Hopefully, the work developed above may ultimately find its way as a valued ingredient in such inverse problem strategies.

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References

[1] Martin, P. A., Rizzo, F. J.: On boundary integral equations for crack problems. Proc. Roy. Soc. A **421**, 341–355 (1989).

[2] Rudolphi, T. J., Krishnasamy, G., Schmerr, L. W., Rizzo, F. J.: On the use of strongly singular integral equations for crack problems. Boundary elements 10 (C. A. Brebbia, ed.), Southampton, England: Springer-Verlag and Computational Mechanics Publications. Sept. 1988.

[3] Jaswon, M. A.: Integral equations methods in potential theory. Proc. Royal. Soc. A **275**, 23–32 (1963).

[4] Banerjee, P. K., Butterfield, P. K.: Boundary element method in engineering. McGraw-Hill 1981.

[5] Mukherjee, S.: Boundary element method in creep and fracture. New York: Applied Sciences Publishers 1982.

[6] Cruse, T. A.: Boundary element analysis in computational fracture mechanics. Boston: Kluwer Academic Publishers 1988.

[7] Hartmann, F.: Introduction to boundary element theory and applications. New York: Springer 1989.

[8] Schenck, H. A.: Improved integral formulation for acoustic radiation problems. J. Acoust. Soc. Am. **44**, 41–58 (1968).

[9] Kleinman, R. E., Roach, G. F.: Boundary integral equations for the three-dimensional Helmholtz equation. SIAM Rev. **16**, 214–236 (1974).

[10] Liu, Y. J., Rizzo, F. J.: A weakly-singular form of the hypersingular boundary integral equation applied to 3-D acoustic wave problems. Computer methods in applied mechanics and engineering. To appear.

[11] Takakuda, K., Koizumi, T., Shibuya, T.: On integral equation methods for crack problems. Bull. J.S.M.E. **28**, 236 (1985).

[12] Nishimura, N., Kobayashi, S.: An improved boundary integral equation method for crack problems. Proc. IUTAM Symposium, San Antonio, Texas, USA, April 13–16, 1987 (T. A. Cruse, ed.), Adv. Boundary Element Method. Berlin–Heidelberg–New York–Tokyo: Springer 1987.

[13] Lin, W., Keer, L. M.: Scattering by a planar three-dimensional crack. J. Acoust. Soc. Am. **82 (4)**, 1442–1448 (1987).

[14] Budreck, D. E., Achenbach, J. D.: Scattering from three-dimensional planar cracks by the boundary integral equation method. J. Appl. Mech. **55**, 405–411 (1988).

[15] Krishnasamy, G., Schmerr, L. W., Rudolphi, T. J., Rizzo, F. J.: Hypersingular boundary integral equations: Some applications in acoustic and elastic wave scattering. J. Appl. Mech. **57**, 404–411 (1990).

[16] Gray, L. J., Martha, L. F., Ingraffea, A. R.: Hypersingular integrals in boundary element fracture analysis. Int. J. of Num. Meth. in Engg. **29**, 1135–1158 (1990).

[17] Muskhelishvili, N. I.: Singular integral equations. Nordhoff: The Netherlands 1953.

[18] Hadamard, J.: Lectures on cauchy's problem in linear partial differential equations. New Haven, Conn.: Yale Univ. Press 1923.

[19] Kutt, H. R.: The numerical evaluation of principal value integrals by finite-part integration. Numer. Math. **24**, 20–210 (1975).

[20] Brando, M. P.: Improper integrals in theoretical aerodynamics: The problem revisited. AIAA J. **25 (9)**, 1258–1260 (1986).

[21] Kaya, A. C., Erdogan, F.: On the solution of integral equations with strongly singular kernels. Q. Appl. Math. **57**, 404–414 (1987).

[22] Ioakimidis, N. I.: A new singular integral equation for the classical crack problem in plane and antiplane elasticity. Int. J. Fracture **21**, 115–122 (1983).

[23] Guiggiani, M., Gigante, A.: A general algorithm for multidimensional cauchy principal value integrals in the boundary element method. J. Appl. Mech. **57**, 906–915 (1990).

[24] Ervin, V. J., Kieser, R., Wendland, W. L.: Numerical approximation of the solution for a model 2-D hypersingular integral equation. Computational engineering with boundary elements, Vol. 1 (S. Gilli, C. A. Brebbia, A. H. D. Cheng, eds.), Computational Mechanics Publications, Southampton 1990.

[25] Guiggiani, M., Krishnasamy, G., Rudolphi, T. J., Rizzo, F. J.: A general algorithm for the numerical solution of hypersingular boundary integral equations. J. Appl. Mech., to appear.

[26] Schwab, C., Wendland, W. L.: Numerical integration of singular and hypersingular integrals in boundary element methods. Numerische Mathematik, in review.

[27] Filippi, P. J. T.: Layer potentials and acoustic diffraction. J. Sound Vibr. **54 (4)**, 473–500 (1977).

[28] Nedelec, J. C.: Integral equations with non integrable kernels. Integral Eq. Operator Th. **5**, 562–572 (1982).

[29] Chien, C. C., Rajiyah, H., Atluri, S. N.: An effective method for solving the hypersingular integral equation in 3-D acoustics. J. Acoust. Soc. Am. **88 (2)**, 918–936 (1990).

[30] Sladek, V., Sladek, J., Balas, J.: Boundary integral formulation of crack problems. Z. Angew. Math. Mech. **66 (2)**, 83–94 (1986).

[31] Bonnet, M.: Regular boundary integral equations for three-dimensional finite or infinite bodies with and without curved cracks in elastodynamics. Boundary element techniques: Applications in engineering, Brebbia, C. A. and Zamani, N. (eds.). Computational Mechanics Publications. Southampton 1989.

[32] Rudolphi, T. J.: The use of simple solutions in the regularization of hypersingular boundary integral equations. Math. Comput. Modelling **15 (3–5)**, 269–278 (1991).

[33] Krishnasamy, G., Rizzo, F. J., Rudolphi, T. J.: Hypersingular boundary integral equations: Their occurrence, interpretation, regularization and computation. To appear in developments in boundary element methods, Vol. 7, Advanced Dynamic Analysis, Elsevier Applied Science Pub.

[34] Krishnaamy, G., Rizzo, F. J., Rudolphi, T. J.: Continuity requirements for density functions in the boundary integral equation method. Computational Mechanics, 9, 267–284 (1992).

[35] Krishnasamy, G., Rizzo, F. J., Liu, Y. J.: Boundary integral equation formulation for thin body problems. In preparation.

[36] Krishnasamy, G., Rizzo, F. J., Liu, Y.: Scattering of acoustic and elastic wave scattering by crack like objects: The role of hypersingular integrals. Review of progress in quantitative nondestructive evaluation (Thompson, D. O., Chimenti, D. E., eds.), Vol. 18. Maine: Plenum Press, Aug. 1991.

[37] Schuster, G. T.: A hybrid BIE + Born series modeling scheme: Generalized Born series. J. Acoust. Soc. Am. **77**, 865–886 (1985).

[38] Kitahara, M., Nakagawa, K.: Born series approach applied to three dimensional elastodynamic inclusion analysis by BIE methods. Proc. Fourth Japan Nat. Symp. on Bound. Elem. Meth., Japan. Soc. Comp. Meth. Engr., Tokyo (1987).

[39] Schafbuch, P. J., Rizzo, F. J., Thompson, R. B.: Elastic wave scattering by multiple inclusions. Proc. ASME Symp. on Enhancing Analysis Techniques for Composite Materials. Winter Annual Meeting, Atlanta, Dec. 1991.

[40] Nishimura, N., Kobayashi, S.: Further applications of regularised integral equations in crack problems. Symp. of the Int. Ass. for Boundary Element Method, Rome, Italy, Oct. 1990.

[41] Kitahara, M., Nakagawa, K.: Phase shift determination of scattered far-fields and its application to an inverse problem. Review of progress in quantitative nondestructive evaluation (Thompson, D. O., Chimenti, D. E., eds.), Vol. 18. Maine: Plenum Press, Aug. 1991.

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