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# Application of Overhauser $C^1$ Continuous Boundary Elements to "Hypersingular" BIE for 3-D Acoustic Wave Problems

Yijun Liu and F. J. Rizzo
*Department of Theoretical and Applied Mechanics, University of Illinois at Urbana-Champaign, 104 S. Wright Street, Urbana, Illinois 61801, USA*

## 1. INTRODUCTION

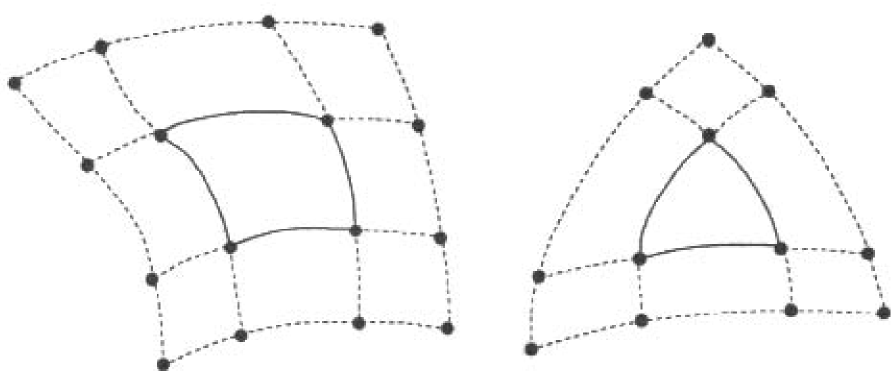
The $C^1$ continuous representations of boundary geometry and variables in the BIE/BEM not only provide more accurate results, but are also demanded by some BIE formulations, such as the "hypersingular" BIE formulations of certain problems [1, 2]. There are several types of $C^1$, or even $C^2$, boundary elements in the literature. Liggett and Salmon [3] introduced cubic spline interpolation in the discretization of BIE formulation. In this interpolation, nodal values of the function and second derivative of the function are used. The nodal values of the second derivative are obtained in terms of the nodal values of the function by solving a separate set of equations. Thus, this method has a global character in the sense that the function inside one element is actually determined by all the nodal values of the function on the entire curve. Watson [4] introduced Hermitian cubic elements in BEM for 2-D problems. The problem with the Hermitian elements is that the tangential derivatives of the function and even cross-derivatives (for 3-D problems) need to be introduced at the nodes. These derivatives are usually not the boundary variables in the conventional BIE formulation and are troublesome to deal with. B-splines have been used in BEM for some time and a more recent work is given by Cabral *et al* [5]. The important character of the B-splines is that the spline curves do not pass through the specified nodes (for geometry) or the nodal values (for function). The geometry (or function) within an element is defined by four control points (or coefficients) multiplied by blending functions. Positions of the control points or values of the coefficients are unknown in advance and need to be determined by solving a system of linear equations (of the same size as that of the BEM system) relating the control points to the nodes or the coefficients to nodal values of the function. Although the matrix of this system has some special features, the solution of this additional system will reduce the efficiency of the method, especially for large real engineering problems. All the above mentioned $C^1$ or $C^2$ boundary elements have been applied only to 2-D problems.

Overhauser $C^1$ continuous line elements for 2-D problems were developed by Ortiz *et al* [6], and surface elements for 3-D problems by Hall and Hibbs [7-9]. The main advantage of the Overhauser elements is that the nodes or nodal values of function are used directly in representing the geometry or function on the elements. No derivatives of the function (as in the cubic splines and Hermitian elements) or some intermediate quantities (such as the control points or coefficients in B-splines) are involved. Thus the definitions of the Overhauser elements are straightforward and easy to program. Numerical results obtained by using Overhauser line elements for 2-D problems [6, 10] clearly show greater accuracy and efficiency compared with the commonly applied $C^0$ elements and other cubic spline elements. Numerical examples of the Overhauser surface (quadrilateral and triangular) elements for 3-D problems, though limited in numbers [7-9], show similar advantages of the Overhauser elements over the usual surface elements regarding accuracy and efficiency.

In this paper, the Overhauser surface elements are applied to 3-D acoustic wave problems for which the "hypersingular" BIE [11] is employed to overcome the fictitious eigenfrequency difficulty of the conventional BIE. This "hypersingular" BIE formulation requires, theoretically, $C^1$ continuity of the density functions [2] in the neighborhood of the source point. Data from numerical experiments involving acoustic scattering problems show that the Overhauser surface elements, in general, can give comparably accurate results with much fewer nodes on the boundary and less computer running time, compared with two types of quadratic elements, namely, conforming quadratic elements and non-conforming quadratic elements. Thus in addition to their high accuracy, the Overhauser surface elements can produce a much smaller system of linear algebraic equations to solve. This is an important feature for the applications of BEM on microcomputers or workstations.

## 2. THE OVERHAUSER ELEMENTS

The Overhauser line element [6, 7] for 2-D problems is defined by four nodes along a curve, where the two inner nodes are at the ends of the element considered and the two outer nodes are on the adjacent elements. Two parabolas are constructed by the first three and last three nodes, respectively. The Overhauser curve is thus formed by a linear blending of the two parabolas. The curve is guaranteed to have inter-element continuous slopes at the nodes. Functions defined on the curve are interpolated in a similar way.



*(a) Quadrilateral element.* *(b) Triangular element.*

*Fig. 1. The Overhauser $C^1$ continuous surface elements.*

The construction of the Overhauser quadrilateral (surface) element [7, 8], Fig.1.(a), is a straightforward generalization of the line element. Sixteen nodes are used in the definition of the quadrilateral element, where four nodes are placed at the corners of the element and twelve others are on the surrounding elements. There are altogether sixteen shape functions employed in the interpolations of the surface and functions defined on the surface. Construction of the Overhauser triangular element [8, 9], Fig.1.(b), on the other hand, is much more complicated. The triangular elements are necessary because the quadrilateral elements alone are not sufficient to produce a mesh which can ensure $C^1$ continuity for all kinds of closed surfaces, [8, 9]. Twelve nodes, three on the corners of the element and nine others on the surrounding (quadrilateral) elements, are used in the definition of the triangular elements. In contrast to the quadrilateral element, the twelve shape functions for the triangular element are lengthy and the derivatives of the shape functions are even more tedious. However, the efficiency in computation of the Overhauser elements will not be hindered by their larger number and lengthy expressions of the shape functions, as will be discussed in the last section. A complete set of the Overhauser $C^1$ continuous surface elements, including the reduced version of the quadrilateral elements for surfaces with corners or edges and the transition elements for imbedding the triangular element in an otherwise quadrilateral mesh, is presented in detail in [8].

The most important feature of the Overhauser elements is that unlike all other $C^1$ or $C^2$ boundary elements, only the nodes or nodal values of functions are employed in the definition of the elements. This same feature is used in the definitions of lower order elements (e.g. linear or quadratic elements). The computer program using lower order elements would keep the same structure if the lower order elements were replaced with the Overhauser elements. No additional work, such as finding the nodal values of the second derivatives for cubic splines [3] and the control points or coefficients for B-splines [5], would be needed except for handling the shape functions. Thus upgrading existing codes with the Overhauser elements is quite straightforward.

The Overhauser quadrilateral and triangular elements developed by Hall and Hibbs [7-9] are applied in this paper. Little modification is made for the triangular element. A new set of side parameters used in the definition of the triangular elements is introduced, which can significantly simplify the expressions of the derivatives of the shape functions. The transition (quadrilateral) elements introduced by Hall and Hibbs are not used. Instead, they are simulated by using the original quadrilateral element, which is done by specifying same coordinates for associated nodes, i.e. treating the transition elements as special cases of the original quadrilateral elements. In this way, one will not need to evaluate and store additional sets of shape functions for the transitional elements. It is tempting for one to simulate the triangular element by the quadrilateral element. However, careful considerations reveal that the inter-element smoothness will be violated by the simulated triangular element.

## 3. THE COMPOSITE BIE FORMULATION

The conventional boundary integral equation for exterior acoustic problems is

$$C(P_o)\phi(P_o) = \int_S \left[ G(P,P_o)\frac{\partial\phi(P)}{\partial n} - \frac{\partial G(P,P_o)}{\partial n}\phi(P) \right] dS(P) + \phi^I(P_o), \qquad \forall P_o \in S, \quad (1)$$

where $\phi$ is the total wave, $\phi^I$ the incident wave (for scattering problems), $G(P,P_o)$ the Green's function for the Helmholtz equation, $n$ the outward normal to the boundary $S$ of the exterior domain $E$, and the coefficient $C(P_o)$ depends on the smoothness of $S$.

The solution of Eq. (1) suffers a nonuniqueness problem when the wavenumber is near or equal to one of the so called fictitious eigenfrequencies [12]. How to circumvent this fictitious eigenfrequency difficulty (FED) for the exterior problems has been a major research area in the applications of BIE to acoustics. The most effective method to deal with the FED is due to Burton and Miller [13]. They proved that the composite BIE formulation, using a linear combination of the conventional BIE (Eq. (1)) and the following "hypersingular" BIE

$$\frac{\partial \phi(P_o)}{\partial n_o} = \int_S \left[ \frac{\partial G(P,P_o)}{\partial n_o} \frac{\partial \phi(P)}{\partial n} - \frac{\partial^2 G(P,P_o)}{\partial n \partial n_o} \phi(P) \right] dS(P) \quad + \quad \frac{\partial \phi^I(P_o)}{\partial n_o}, \qquad P_o \to S, \quad (2)$$

where $n_o$ is the outward normal at $P_o \in S$, can provide unique solutions at all wavenumbers. The major difficulty in implementing this composite BIE formulation has been the treatment of the hypersingular integral in Eq. (2). A recently proposed approach [11] to deal with this hypersingular integral is to transform Eq. (2), by employing some integral identities established in [14] for the static Green function $\overline{G}(P,P_o)$, into the following weakly-singular form,

$$\begin{aligned}
&\frac{\partial \phi(P_o)}{\partial n_o} + \int_S \frac{\partial^2 \overline{G}(P,P_o)}{\partial n \partial n_o} [\phi(P) - \phi(P_o) - \phi_{,k}(P_o)(x_k - x_{ok})] dS(P) \\
&\quad + \int_S \frac{\partial^2}{\partial n \partial n_o} [G(P,P_o) - \overline{G}(P,P_o)] \, \phi(P) dS(P) \\
&\quad = \int_S \frac{\partial G(P,P_o)}{\partial n_o} [\phi_{,k}(P) - \phi_{,k}(P_o)] n_k(P) dS(P) \\
&\quad + \int_S \frac{\partial}{\partial n_o} [G(P,P_o) - \overline{G}(P,P_o)] \, [\phi_{,k}(P_o) n_k(P)] dS(P) \\
&\quad + \frac{\partial \phi^I(P_o)}{\partial n_o}, \qquad\qquad \forall P_o \in S, \qquad (3)
\end{aligned}$$

where $x_k$ and $x_{ok}$ are coordinates of $P$ and $P_o$, respectively. All the integrals in Eq. (3) are at most weakly singular and the commonly used quadratures for conventional BIE are sufficient to compute them. However, there is a theoretical restriction on the density function in either Eq. (2) or Eq. (3). For the hypersingular integral in Eq. (2) to exist or for the procedures leading to Eq. (3) to be valid, the density function $\phi$ must be $C^1$ continuous, at least at the source point $P_o$, see e.g. [1,2,11]. This restriction on the "hypersingular" BIE will demand, theoretically, $C^1$ boundary elements in the discretization of Eq. (3).

The Overhauser elements will be applied to the composite BIE formulation, i.e. the linear combination of Eq. (1) and Eq. (3), in the next section. For comparison, two types of quadratic boundary elements are also applied, namely, conforming and non-conforming quadratic elements (quadrilateral and triangular). The conforming quadratic elements are the commonly used eight node quadrilateral and six node triangular elements in the BEM literature. These elements are in the $C^0$ element category and violate the smoothness requirement for Eq. (3). Nevertheless, if the smoothness requirement is relaxed in some sense and certain techniques are employed to handle the nonuniqueness of the gradient of $\phi$ at the source point, good results can be obtained by using these conforming elements [11]. However, the validity of applying the conforming quadratic elements to "hypersingular" BIE's is still an open question. The non-conforming quadratic elements are obtained by simply moving the nodes some distance inside the elements. Hence the $C^1$ continuous requirement on the density function is satisfied in the neighborhood of the source point.

4. NUMERICAL RESULTS

The scattering problem of a plane incident wave $\phi^I$ from a rigid sphere ($\partial\phi/\partial n = 0$ on the boundary) of radius $a$ is considered here. The magnitudes of the ratios of the scattering wave $\phi^S$ to $\phi^I$ at a radius $R=5a$ are plotted versus the angle $\theta$ between the direction of the incident wave and $\boldsymbol{R}$. In all the cases, $M$ is the total number of elements on the sphere and $N$ the number of nodes. Two plots of the Overhauser element meshes for the sphere are shown in Fig. 2.

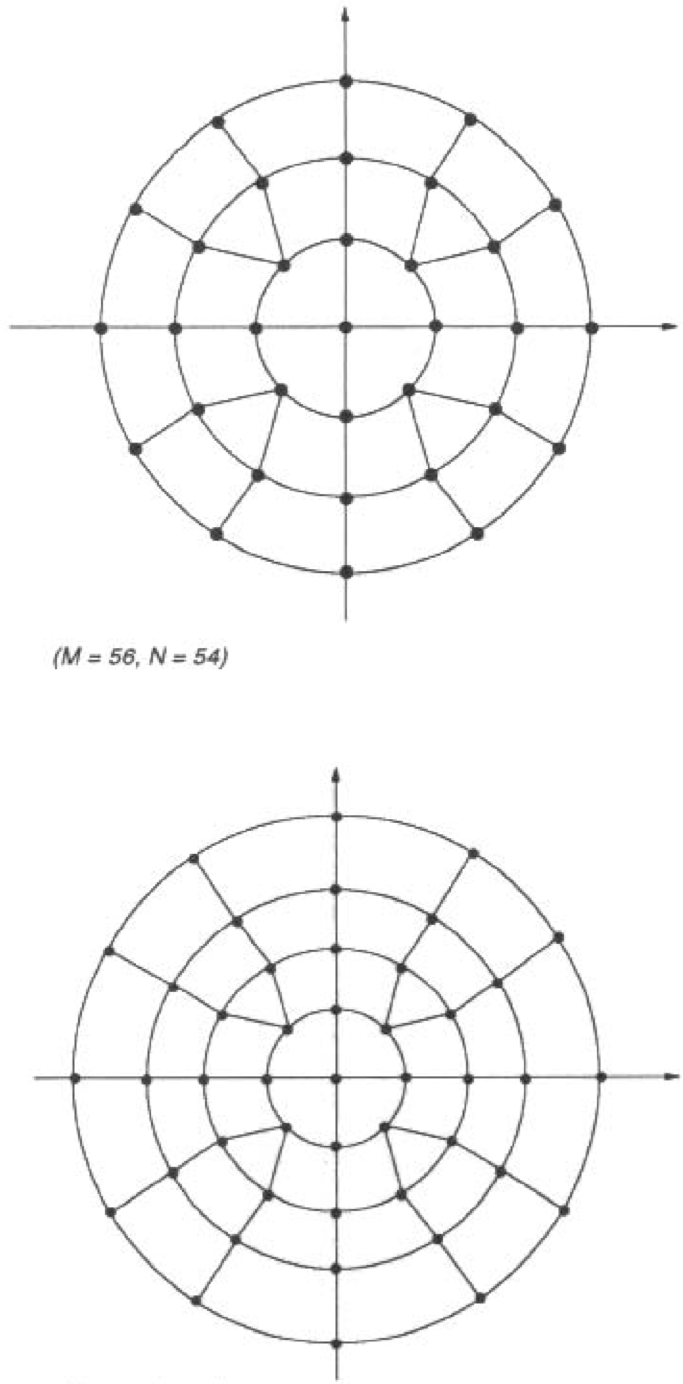
All the BEM results reported here were obtained by applying the composite BIE formulation mentioned in the previous section and the wavenumbers studied are fictitious eigenfrequencies at which the conventional BIE formulation cannot provide unique solutions (indicated by a large condition number of the coefficient matrix). The computation was performed on an Apollo 10000 machine.

Figure 3 shows the convergence of the Overhauser elements at wavenumber $ka = 2\pi$. The convergence of the results is observed as the number of elements increase. It is noticed that results on the shadow side ($\theta = 0$ degree) converge at a slower rate than on the illuminated side ($\theta = 180$ degree, backscattering direction). This is a typical phenomenon for the BEM solutions of this problem.

Figure 4 is a comparison of the conforming quadratic, non-conforming quadratic and Overhauser elements, at $ka = \pi$, where the numbers of elements are fixed at $M = 56$. The numerical values at the shadow side by the Overhauser elements are not as good as the conforming and non-conforming quadratic elements, but the computer running time for the Overhauser elements is much less than those for the latter two. Notice that the size of the system of equations (*N by N*) for the Overhauser elements is only about *1/3* of that for the conforming elements and *1/8* of that for non-conforming elements.

Figure 5 is a comparison of the three types of elements, at $ka = \pi$, where about the same numbers of nodes are used for the three elements. This is probably a more important comparison since number of nodes used is closely related to the size of the system of equations and usually is chosen as the parameter for comparison of different elements in BEM literature, e.g. [3]. It is shown that the best results are achieved by the Overhauser elements in this case. The results by the non-conforming elements are unacceptable due to the small number of elements which can be generated from the given number of nodes. However, the running time for Overhauser elements is the longest in this case, while for the non-conforming elements the shortest. The running time (mainly the system formation time plus the solution time) for the three types of elements will be discussed in more detail in the next section.

Figure 6 is a more realistic comparison of the three elements, at $ka = 2\pi$. First, the non-conforming elements were tested by using meshes of increasing numbers of elements (or nodes) until reasonably good results were obtained. Then, the conforming and Overhauser elements were tested in the same way until the results of about the same accuracy as that by the non-conforming elements were achieved. The results by the final meshes of the three elements were plotted and the running time recorded. It is observed that for the conforming elements, about the same number of nodes as that for the non-conforming elements is needed to achieve about the same accuracy, while for the Overhauser elements, the number of nodes needed is only nearly *1/3* of those for the other two types of elements. More importantly, the running time of the Overhauser elements is the shortest, about *3/4* of that of the non-conforming elements and *1/3* of that of the conforming elements.



(M = 56, N = 54)

(M = 80, N = 78)

*Fig. 2. Overhauser element meshes for the sphere.*

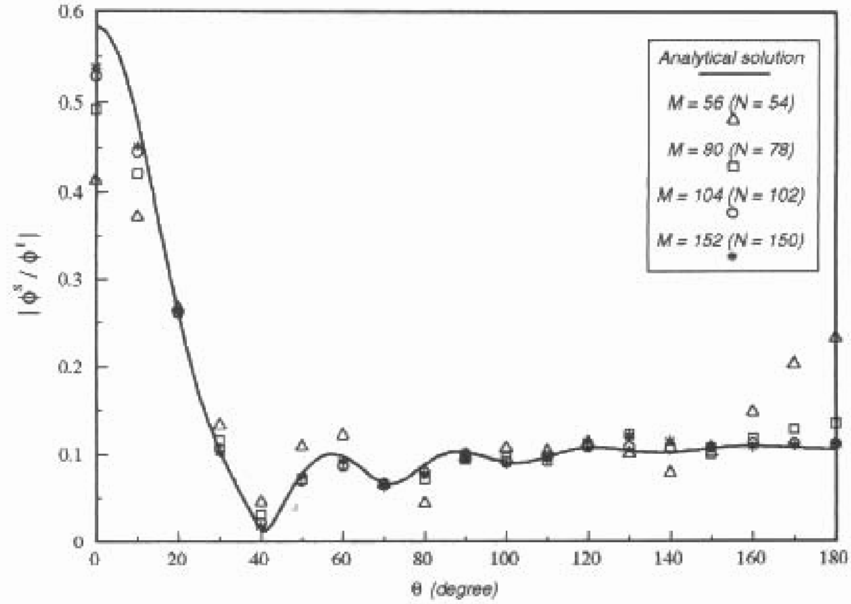



*Fig. 3. Convergence of the Overhauser elements at wavenumber ka = 2π.*

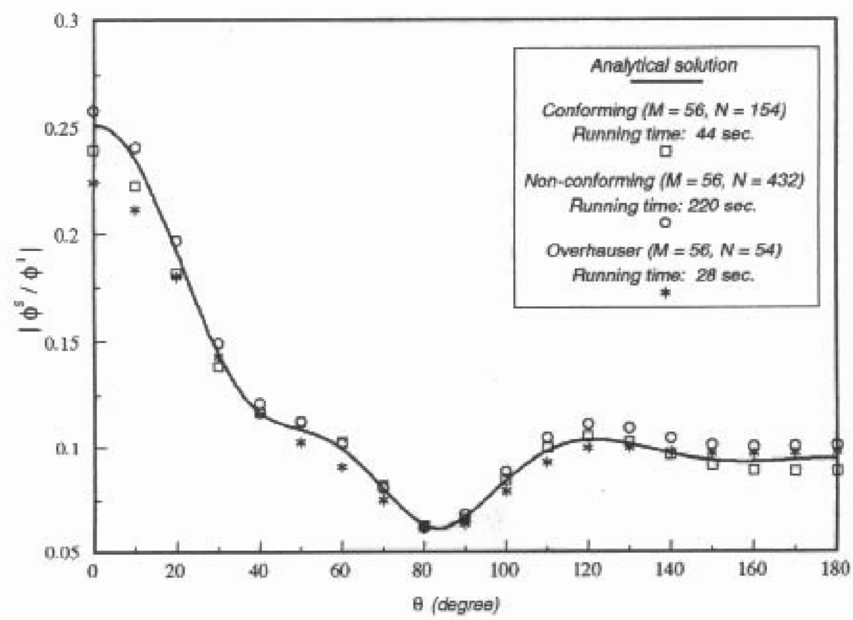



*Fig. 4. Comparison at wavenumber ka = π, for a fixed number of elements.*

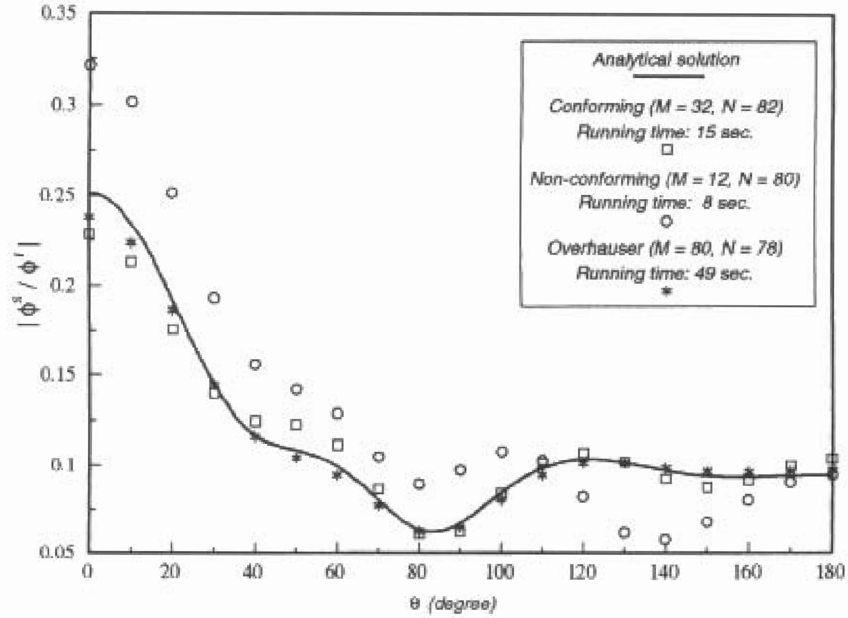



*Fig. 5. Comparison at wavenumber $ka = \pi$, for a 'fixed' number of nodes.*

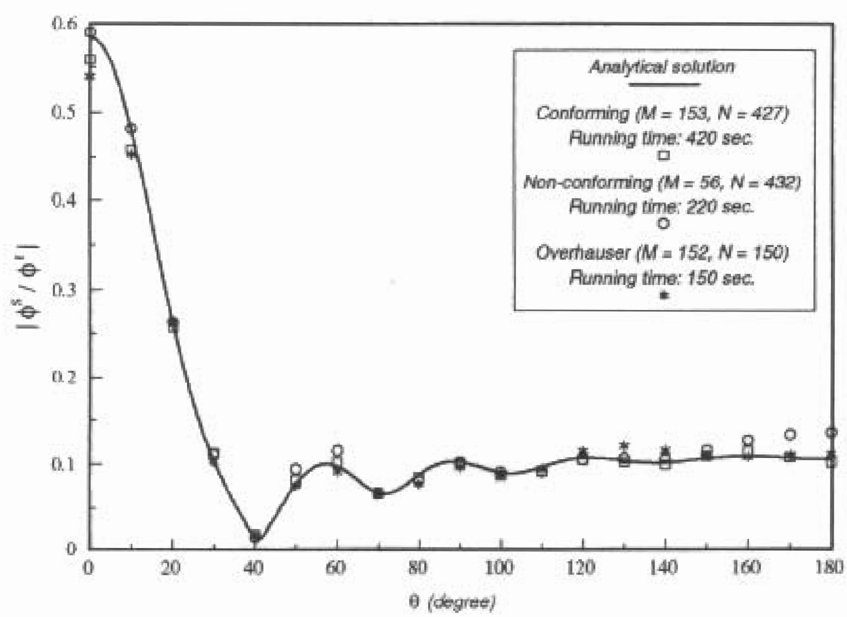



*Fig. 6. Comparison at wavenumber $ka = 2\pi$.*

## 5. DISCUSSIONS

The high accuracy of the Overhauser surface elements is further demonstrated in this paper by the numerical examples of acoustic wave problems. To achieve the same level of accuracy, considerably fewer nodes can be employed for the Overhauser elements than for quadratic elements.

One might expect that the computer time for the Overhauser elements would be longer than for the quadratic elements. Considering the large number and lengthy expressions of the shape functions for the Overhauser elements, this will be true, at least for setting up the system (formation time) as stated in [7]. However, this will be changed if the shape functions and their derivatives are evaluated only *once* and then stored, as is done in the computer codes used for this comparison study. In this way, the computation of shape functions would not be a factor in the running time for solving a problem. The formation time is then proportional to the factor $N \times M \times S$, where $N$ is the number of nodes, $M$ the number of elements and $S$ the number of shape functions (equal to the number of summations performed for the integration on an element). Suppose that most of the elements used are quadrilateral ones, then $N = 3M$, $8M$ *and* $M$, approximately, for the conforming quadratic, non-conforming quadratic and Overhauser elements, respectively. Thus the ratios of the formation time for the three types of elements are *1.5:4:1* for a fixed number of elements or *2.67:1:16* for a fixed number of nodes. Test results show that these estimates for the formation time hold. The second estimate (for a fixed number of nodes) is not in favor of the Overhauser elements. However, one need not use the same number of nodes for the Overhauser elements to achieve a given accuracy, as shown by the numerical examples, and perhaps one needs only half the number of nodes compared to the quadratic elements. Therefore, the solution time for the Overhauser elements will be much less. For problems of a moderate size (about a few hundred nodes), the solution time will be longer than the formation time. Thus total running time will be in favor of the Overhauser elements and this is even more obvious for larger size problems.

Mesh generation of the Overhauser elements is not as easy as those of the quadratic elements because of the relatively complicated connectivity of the Overhauser elements. To apply the Overhauser elements to real engineering problems, special software for mesh generation will be needed.

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