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Finite Deflection Analysis of Heated Elastic Plates by the Boundary Element Method

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ABSTRACT

This paper presents the formulation for the finite deflection analysis of heated elastic plates by the boundary element method (BEM). A relaxation iterative technique is applied to obtain the numerical solutions.

1.INTRODUCTION

Very few applications of the BEM to the geometrically nonlinear analysis of plates have been reported. Tanaka [1],Kamiya et al.[2],[3] and Ye and Liu [4] are known to have published work in this field. This paper, as a further development of the previous work [4], gives an analysis of the large deflection problems of thermoelastic plates by the BEM.

2.GOVERNING DIFFERENTIAL EQUATIONS AND RELATIONS

Let us denote by Ω a two dimensional domain enclosed by the boundary Γ with Cartesian coordinates x_1 and x_2 which represent the median plane of a flate elastic plate, with z denoting the distance from this plane (see Fig.1). The thickness of the plate is h. The displacements, in the x_1, x_2 and z directions, of points on the middle surface are denoted by u_1, u_2 and w, respectively. Let $T(x_1, x_2, z)$ be the temperature distribution, α the coefficient of the linear thermal expansion, ν the Poisson's ratio and E the Young's modulus.

The bending and twisting moments are

$$M_{ij}=-D_{ijlm}w_{,lm}+M_T\delta_{ij} \quad (1)$$

where δ_{ij} is the Kronecker symbol and $i,j,l,m=1,2$.
The rigidity tensor is

$$D_{ijlm}=D[\nu\delta_{ij}\delta_{lm}+(1-\nu)\delta_{il}\delta_{jm}]$$

where $D=Eh^3/12(1-\nu^2)$.
The symbol M_T denotes the quantity

$$M_T=-\frac{E\alpha}{1-\nu}\int_{-h/2}^{h/2}T(x_1,x_2,z)zdz$$

The shear forces may be represented as

$$Q_i=M_{ij,j}=-D_{ijlm}w_{,lmj}+M_{T,i} \quad (2)$$

On the boundary, the bending and twisting moments are

$$M_n=-D(w_{,nn}+\nu w_{,tt})+M_T \quad (3)$$

$$M_{nt}=-D(1-\nu)w_{,nt} \quad (4)$$

where n and t are the outward normal and tangent at the boundary, respectively.
The shear forces V_n and the Kirchhoff equivalent shear force K_n on the boundary in the small deflection case are

$$V_n=Q_in_i \quad (5)$$

$$K_n=V_n+M_{nt,s} \quad (6)$$

where n_i $(i=1,2)$ are direction cosines of the normal n.

The governing nonlinear differential equations for the finite deflection problems of the thermoelastic plates are

$$D\nabla^4w=\bar{q}+h\sigma_{ij}w_{,ij}+\nabla^2M_T \quad (7a)$$

$$\sigma_{ij,j}=0 \quad (7b)$$

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}+w_{,i}w_{,j}) \quad (7c)$$

$$\sigma_{ij}=E_{ijlm}(e_{lm}-\alpha T_0\delta_{lm}) \quad (7d)$$

where

$$T_0=\frac{1}{h}\int_{-h/2}^{h/2}T(x_1,x_2,z)dz$$

and E_{ijlm} is the elastic modulus tensor, e_{ij} and σ_{ij} are the in-plane strains and stresses, respectively.

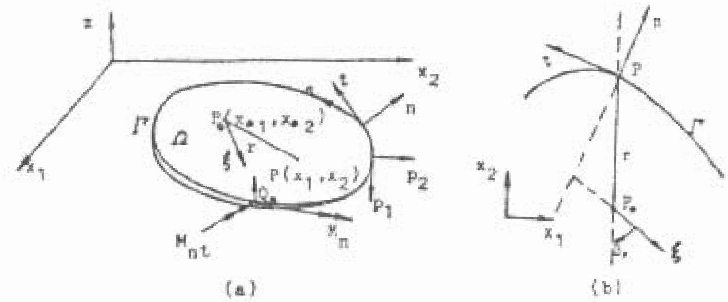


Fig.1 Notations

3. INTEGRAL EQUATIONS

Starting with the equation (7a) and making use of the generalized Green identity one can obtain the following integral equations

$$\int_\Omega qw^*d\Omega+\int_\Omega M_T\nabla^2w^*d\Omega+$$

$$\int_\Gamma(w^*K_n-wK_n^*+w_{,n}M_n^*-w_{,n}^*M_n)d\Gamma$$

$$=\begin{cases}Dw(P_0)\ , & P_0\in\Omega \quad (8a)\\ \frac{1}{2}Dw(P_0)\ , & P_0\in\Gamma \quad (8b)\end{cases}$$

where $q=\bar{q}+h\sigma_{ij}w_{,ij}$ and

$$w^*(P,P_0)=\frac{1}{8\pi}r^2\ln r \quad (9)$$

is the fundamental solution of the biharmonic equation; M_n^* and K_n^* are determined by the substitution of $w=w^*$ into the formulae (3) with $M_T=0$ and (6) respectively.
Differentiation of the equations (8) with respect to ξ (see Fig.1) provides

$$\int_\Omega qw^*_{,\xi}d\Omega+\int_\Omega M_T(\nabla^2w^*)_{,\xi}d\Omega+$$

$$\int_\Gamma(w^*_{,\xi}K_n-wK^*_{n,\xi}+w_{,n}M^*_{n,\xi}-w^*_{,n\xi}M_n)d\Gamma$$

$$=\begin{cases}Dw_{,\xi}(P_0)\ , & P_0\in\Omega \quad (10a)\\ \frac{1}{2}Dw_{,\xi}(P_0)\ , & P_0\in\Gamma \quad (10b)\end{cases}$$

where $(\nabla^2w^*)_{,\xi}=\cos\beta_0/2\pi r$ and the angle β_0 is defined in Fig.1(b).
Consequently, a set of boundary integral equations, consisting of (8b) and (10b), is now formulated which corresponds to the nonlinear bending deformation including unknown components of the deflection,

shear force on the boundary.

The second domain integral on the left-hand side of the equation (10a), according to Mikhlin[5], is a singular integral with weak singularity which can not be differentiated directly under the integral sign. The proper expression for the derivative of this singular integral can be written as

$$\frac{\partial}{\partial x_j}\int_\Omega M_T(\nabla^2 w^*)_{,i}\,d\Omega=\int_\Omega M_T(\nabla^2 w^*)_{,ij}\,d\Omega+\frac{1}{2}M_T(P_\circ)\delta_{ij} \qquad (11)$$

where

$$(\nabla^2 w^*)_{,ij}=\frac{1}{2\pi r^2}(\delta_{ij}-2r_{,i}r_{,j}) \qquad (12)$$

and $(\ )_{,i}=(\ )/x_{\circ i}$. Thus the differentiation of both sides of the equation (10a) with respect to the coordinates $x_{\circ j}$ of the source point $P_\circ$ leads to

$$Dw_{,ij}(P_\circ)=\int_\Omega q w^*_{,ij}\,d\Omega+\int_\Omega M_T(\nabla^2 w^*)_{,ij}\,d\Omega+\frac{1}{2}M_T(P_\circ)\delta_{ij}+\int_\Gamma(w^*_{,ij}K_n-wK^*_{n,ij}+w_{,n}M^*_{n,ij}-w^*_{,nij}M_n)\,d\Gamma,\quad P_\circ\in\Omega \qquad (13)$$

The formulae (8a), (10a) and (13) can be used to evaluate the deflection, rotation, bending and twisting moments at any point inside the plate when the boundary unknowns have been determined.

By virtue of the generalized Green identity, one can obtain from the equations (7b),(7c) and (7d) the following integral equations which correspond to the membrane deformation of the plate

$$\int_\Gamma p_k u^*_{ik}\,d\Gamma-\int_\Gamma p^*_{ik}u_k\,d\Gamma-\frac{1}{2}\int_\Omega \sigma^*_{ikl}w_{,k}w_{,l}\,d\Omega+\int_\Omega \sigma^*_{ikk}\alpha T_\circ\,d\Omega=\begin{cases}u_i(P_\circ), & P_\circ\in\Omega \quad (14a)\\ \frac{1}{2}u_i(P_\circ), & P_\circ\in\Gamma \quad (14b)\end{cases}$$

where u^*_{ik} and p^*_{ik} are the fundamental solutions of the governing equations of the two-dimensional elasticity given in [6] p.141 and p.142; and

$$\sigma^*_{ikl}=\frac{1}{4\pi(1-\bar\nu)r}\left[2r_{,i}r_{,k}r_{,l}+(1-2\bar\nu)(\delta_{ik}r_{,l}+\delta_{il}r_{,k}-\delta_{kl}r_{,i})\right] \qquad (15)$$

here $\bar\nu=\nu/(1+\nu)$.

Differentiating (14a) with respect to $x_{\circ i}$ and taking into account of the derivative of the singular integral with weak singularities one obtains the expression of $u_{i,j}$ in the domain. Substitution of this expression into the geometric relations (7c) and the constitutive relations (7d) provides the following expression for membrane stresses inside the plate

$$\sigma_{ij}(P_\circ)=\int_\Gamma D_{ijk}p_k\,d\Gamma-\int_\Gamma S_{ijk}u_k\,d\Gamma-\frac{1}{2}\int_\Omega T_{ijkl}w_{,k}w_{,l}\,d\Omega+\int_\Omega H_{ij}\alpha T_\circ\,d\Omega+\frac{G}{8(1-\bar\nu)}(2w_{,i}w_{,j}+w_{,k}w_{,k}\delta_{ij})-\frac{G\alpha T_\circ}{1-\bar\nu}\delta_{ij}\qquad P_\circ\in\Omega \qquad (16)$$

where the expressions for D_{ijk} and S_{ijk} can be found in [6], p.142, and

$$T_{ijkl}=\frac{\phi_{ijkl}(P,P_\circ)}{r^2} \qquad (17)$$

$$H_{ij}=\frac{G}{\pi(1-\bar\nu)r^2}(\delta_{ij}-2r_{,i}r_{,j}) \qquad (18)$$

where $\phi_{ijkl}(P,P_\circ)$ is given in the Appendix.

It is not difficult to prove that the functions T_{ijkl} and H_{ij} satisfy the condition for the existance of the Cauchy principle values of the related singular integrals.

Thus, the formulation of the boundary integral equations for the finite deflection analysis of heated elastic plates has been accomplished. These equations will be solved numerically by the BEM with a relaxation iterative technique which has been developed in our previous work [4].

4. NUMERICAL EXAMPLES

The applicability and some advantages of the present approach are briefly illustrated by the following examples.

Figure 2 shows the results obtained for a clamped circular plate under uniform lateral load. The results are compared with the analytical solutions [7], and the agreement between the two solutions is perfect for 20 constant boundary elements.

Figure 3 shows a circular plate subjected to the non-uniform temperature distribution

$$T(r,z)=[T_\circ+T_1(1-r^2/a^2)](1+2z/3h)$$

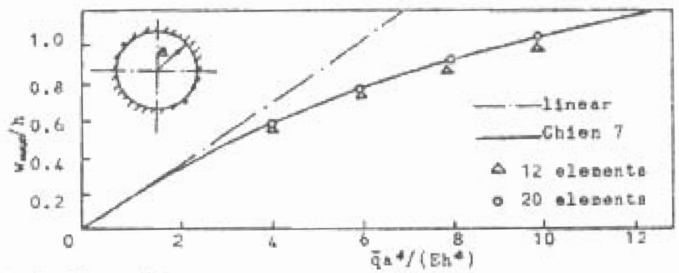


Fig.2 Clamped Circular Plate Under Uniform Lateral Load $\bar{q}$

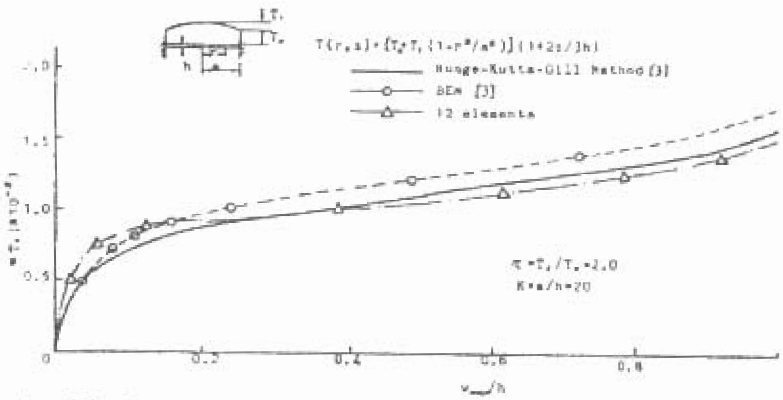
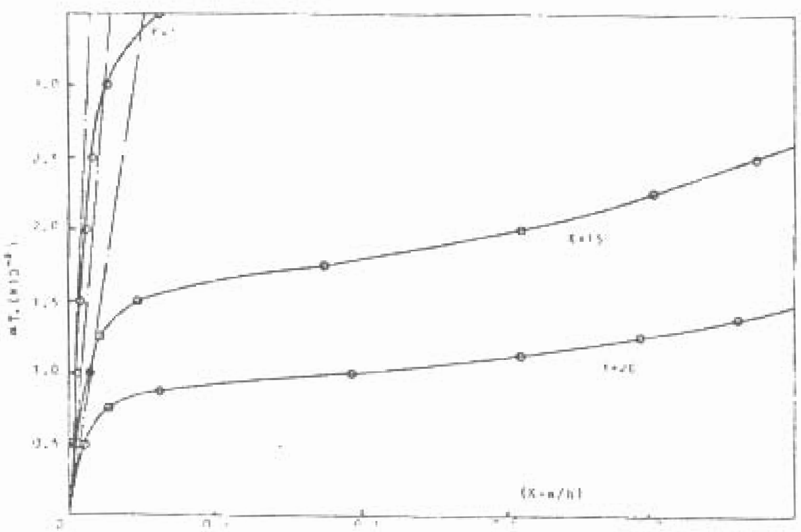


Fig. 3 Circular Plate Under Thermal Load



where $r=(x_1^2+x_2^2)^{1/2}$, T_0 and T_1 are constants. The plate is clamped at the edge. The results obtained by using 12 constant boundary elements for $\tau=T_1/T_0=2.0$ are portrayed. In comparision with the boundary element solution based on the Berger equation [3], the present results generally are in better agreement with the solution by the Runge-Kutta-Gill method also given in [3].

Figure 4 illustrates the maximum deflection of the plate in the above example vs. the temperature distribution with different ratio of the plate radius to the plate thickness K=a/h.

5.CONCLUSION

The present paper shows that the boundary element method can be employed to solve the finite deflection problems of the heated thin elastic plates with satisfactorily accurate solutions.

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APPENDIX

The expression of $\phi_{ijkl}(P,P_0)$ in formula (17) is given below

$$\phi_{ijkl}=-\frac{G}{2\pi(1-\bar{\nu})}\left[8r_{,i}r_{,j}r_{,k}r_{,l}+(1-4\bar{\nu})\delta_{ij}\delta_{kl}-(1-2\bar{\nu})(2\delta_{ij}r_{,k}r_{,l}+2\delta_{kl}r_{,i}r_{,j}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})-2\bar{\nu}(\delta_{ik}r_{,j}r_{,l}+\delta_{jk}r_{,i}r_{,l}+\delta_{il}r_{,j}r_{,k}+\delta_{jl}r_{,i}r_{,k})\right]$$

Spline Boundary Element Method for Shallow Thin-shells*

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ABSTRACT

In this paper, we do not obtain the fundamental solutions for shallow shells directly, but first by reducing the problem to the analysis of two equivalent thin-plates, then, by coupling the solutions of spline boundary element method for the two plates. The results with satisfactory précision can be obtained even when fewer degrees of freedom are implemented. All the unknowns, both at interior points and on boundaries, can be calculated simultaneously and both linear and non-linear analysis for shells and plates can also be worked out in a single programm.

1. INTRODUCTION

It is rather complicated to find the fundamental solutions of shallow shells.[1][2] Instead of using the ordinary direct approach, the authors simplified the bending problem of shallow shells to analysis of two thin-plates. Satisfactory results can be observed by employing spline boundary element method for thin-plate.

2. BASIC PRINCIPLE

For a shallow shell whose mid-surface equation is $Z=\frac{1}{2}(k_x x^2+k_y y^2)$. The governing equations under normal loads q can be written as

$$\begin{aligned} D\nabla^2\nabla^2 W &= L(W,\varphi)+\nabla_k^2\varphi+q \\ \nabla^2\nabla^2\varphi &= -Et\left[\tfrac{1}{2}L(W,W)+\nabla_k^2 W\right] \end{aligned} \qquad (2.1)$$

In which

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \qquad \nabla_k^2=k_y\frac{\partial^2}{\partial x^2}+k_x\frac{\partial^2}{\partial y^2},$$

t —— thickness of shell

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