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BEM for solving problems governed by Helmholtz equations

An overview

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Content

PDE for acoustics

- Ideal, compressible fluid governed by the sound pressure p

$$\begin{aligned}
\nabla^2 p(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} p(\mathbf{x},t) &= 0 && (\mathbf{x},t) \in \Omega \times (0,\infty) \\
p(\mathbf{y},t) &= g_D(\mathbf{y},t) && (\mathbf{y},t) \in \Gamma_D \times (0,\infty) \\
q(\mathbf{y},t) = (\mathcal{T}p)(\mathbf{y},t) = \frac{\partial p}{\partial n} &= g_N(\mathbf{y},t) && (\mathbf{y},t) \in \Gamma_N \times (0,\infty) \\
p(\mathbf{x},0) = \frac{\partial p}{\partial t} &= 0 && (\mathbf{x},t) \in \Omega \times (0)
\end{aligned}$$

  in the domain Ω with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, and the speed of sound c
- Viscous fluid

$$\nabla^2 p(\mathbf{x},t) - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}(\mathbf{x},t) - \frac{R}{\rho c^2}\frac{\partial p}{\partial t}(\mathbf{x},t) = 0$$

  with the flow resistance R
- Wave number

$$k = \frac{\omega}{c} \qquad \text{for viscous fluids} \quad \bar{k} = k\left(1 + \mathrm{i}\frac{R}{2\rho\omega}\right) = k + \mathrm{i}\mu$$

  with the density ρ and the absorption coefficient μ

Fundamental solutions

- Ideal, compressible fluid

$$G(\mathbf{x},t|\mathbf{y},\tau) = \frac{1}{4\pi r}\delta\left(t - \tau - \frac{r}{c}\right)$$

  with $r = |\mathbf{x} - \mathbf{y}|$ and the Dirac distribution $\delta(x)$
- Viscous fluid

$$G(\mathbf{x},t|\mathbf{y},\tau) = \frac{e^{\alpha\frac{t'}{2}}}{4\pi r}\left[\delta\left(t' - \frac{r}{c}\right) + \frac{\alpha r}{2\sqrt{(t')^2 - \frac{r^2}{c^2}}} I_1\left(\frac{\alpha}{2}\sqrt{(t')^2 - \frac{r^2}{c^2}}\right) H\left(t' - \frac{r}{c}\right)\right]$$

  with the modified Bessel function of first kind $I_1(x)$ and $t' = t - \tau$
- Half space solution

$$\begin{aligned}
\text{for } q = 0: \quad G^H(\mathbf{x},t|\mathbf{y},\tau) &= G(\mathbf{x},t|\mathbf{y},\tau) + G(\mathbf{x}',t|\mathbf{y},\tau) \\
\text{for } p = 0: \quad G^H(\mathbf{x},t|\mathbf{y},\tau) &= G(\mathbf{x},t|\mathbf{y},\tau) - G(\mathbf{x}',t|\mathbf{y},\tau)
\end{aligned}$$

  with the *mirror* point $\mathbf{x}'$
- Laplace/Fourier domain

$$\hat{G}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi r}e^{kr} \qquad \text{or} \quad \bar{G}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi r}e^{-\mathrm{i}kr}$$

Integral equations

- Representation formula in time domain

$$p(\mathbf{x},t) = \int_0^t \int_\Gamma \left[G(\mathbf{x}-\mathbf{y},t-\tau)\, q(\mathbf{y},\tau) - \frac{\partial G}{\partial \mathbf{n}}(\mathbf{x}-\mathbf{y},t-\tau)\, p(\mathbf{y},\tau) \right] \mathrm{d}\Gamma_y \, \mathrm{d}\tau$$

- Boundary integral equation in time domain

$$4\pi c(\mathbf{x})\, p(\mathbf{x},t) = \int_\Gamma q\left(\mathbf{y},t-\frac{r}{c}\right) \frac{1}{r} \,\mathrm{d}\Gamma_y + \int_\Gamma p\left(\mathbf{y},t-\frac{r}{c}\right) \frac{1}{r^2}\frac{\partial r}{\partial \mathbf{n}} \,\mathrm{d}\Gamma_y$$
$$+ \int_\Gamma \frac{\partial}{\partial t} p\left(\mathbf{y},t-\frac{r}{c}\right) \frac{1}{rc}\frac{\partial r}{\partial \mathbf{n}} \,\mathrm{d}\Gamma_y$$

with an analytical integration in time

- Boundary integral equation in Laplace/Fourier domain

$$c(\mathbf{x})\hat{p}(\mathbf{x}) + \int_\Gamma \hat{p}(\mathbf{y}) \frac{\partial \hat{G}(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}} \,\mathrm{d}\Gamma_y = \int_\Gamma \hat{q}(\mathbf{y})\, \hat{G}(\mathbf{x},\mathbf{y}) \,\mathrm{d}\Gamma_y$$

Integral equation in operator notation

- 1^{st} Boundary integral equation

$$(\mathcal{V} * q)(\mathbf{x},t) = \mathcal{C}(\mathbf{x})p(\mathbf{x},t) + (\mathcal{K} * p)(\mathbf{x},t) \qquad (\mathbf{x},t) \in \Gamma \times (0,\infty)$$

- 2^{nd} Boundary integral equation

$$(\mathcal{D} * p)(\mathbf{x},t) = (\mathcal{I} - \mathcal{C}(\mathbf{x}))\, q(\mathbf{x},t) - (\mathcal{K}' * q)(\mathbf{x},t) \qquad (\mathbf{x},t) \in \Gamma \times (0,\infty)$$

- Operators

$$(\mathcal{V} * q)(\mathbf{x},t) = \int_0^t \int_\Gamma G(\mathbf{x}-\mathbf{y},t-\tau) q(\mathbf{y},\tau) \,\mathrm{d}s_\mathbf{y} \,\mathrm{d}\tau$$
$$\mathcal{C}(\mathbf{x}) = \mathcal{I} + \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(\mathbf{x}) \cap \Omega} (\mathcal{T}_\mathbf{y} G)(\mathbf{x}-\mathbf{y},0) \,\mathrm{d}s_\mathbf{y}$$
$$(\mathcal{K} * p)(\mathbf{x},t) = \lim_{\varepsilon \to 0} \int_0^t \int_{\Gamma \setminus B_\varepsilon(\mathbf{x})} (\mathcal{T}_\mathbf{y} G)(\mathbf{x}-\mathbf{y},t-\tau) p(\mathbf{y},\tau) \,\mathrm{d}s_\mathbf{y} \,\mathrm{d}\tau$$
$$(\mathcal{K}' * q)(\mathbf{x},t) = \lim_{\varepsilon \to 0} \int_0^t \int_{\Gamma \setminus B_\varepsilon(\mathbf{x})} (\mathcal{T}_\mathbf{x} G)(\mathbf{x}-\mathbf{y},t-\tau) q(\mathbf{y},\tau) \,\mathrm{d}s_\mathbf{y} \,\mathrm{d}\tau$$
$$(\mathcal{D} * p)(\mathbf{x},t) = -\lim_{\varepsilon \to 0} \int_0^t \mathcal{T}_\mathbf{x} \int_{\Gamma \setminus B_\varepsilon(\mathbf{x})} (\mathcal{T}_\mathbf{y} G)(\mathbf{x}-\mathbf{y},t-\tau) p(\mathbf{y},\tau) \,\mathrm{d}s_\mathbf{y} \,\mathrm{d}\tau$$

Source terms

- Multiple sound sources at discrete points ξ_j with intensity $A_j(t)$ yield an additional term in the integral equation

$$\sum_{j=1}^{q_{num}} A_j \left(t - \frac{|\mathbf{x} - \xi_j|}{c} \right) \frac{1}{|\mathbf{x} - \xi_j|}$$

- Moving sound sources (without volume) can be treated by the Helmholtz equation for the velocity potential in a moving coordinate system using a different fundamental solution

$$G_m(\mathbf{x}, t; \mathbf{y}, \tau) = \frac{\delta(t - \tau - r/c)}{4\pi[1 - M_R]}$$

with $M_R = \dfrac{\mathbf{v} \cdot \mathbf{r}}{rc}$

Irregular frequencies

- Non-uniqueness for exterior problems, e.g., scattering
- Burton-Miller approach (also Brakhage and Werner)

$$(\mathcal{V}q)(\mathbf{x}, t) + \alpha(\mathcal{D}p)(\mathbf{x}, t) = \mathcal{C}(\mathbf{x})p(\mathbf{x}, t) + (\mathcal{K}p)(\mathbf{x}, t) + \alpha\left[(\mathcal{I} - \mathcal{C}(\mathbf{x}))\, q(\mathbf{x}, t) - (\mathcal{K}'q)(\mathbf{x}, t)\right]$$

$$(\mathbf{x}, t) \in \Gamma \times (0, \infty)$$

- The factor α can be chosen arbitrarily but often

$$\alpha = \frac{\mathrm{i}}{k}$$

- For small frequencies the CHIEF-method can also be used
- There exist techniques with modified fundamental solutions

Collocation and Galerkin method

- Collocation method - 1st integral equation is used and solved at distinct points. Collocation points usually are the nodal values
- Galerkin method
  Introduction of arbitrary but fixed extensions, $\tilde{\mathbf{g}}_D$ and $\tilde{\mathbf{g}}_N$,

$$
\begin{aligned}
p &= \tilde{p} + \tilde{g}_D \quad \text{with} \quad \tilde{g}_D(\mathbf{x},t) = g_D(\mathbf{x},t) \quad (\mathbf{x},t) \in \Gamma_D \times (0,\infty) \\
q &= \tilde{q} + \tilde{g}_N \quad \text{with} \quad \tilde{g}_N(\mathbf{x},t) = g_N(\mathbf{x},t) \quad (\mathbf{x},t) \in \Gamma_N \times (0,\infty)
\end{aligned}
$$

yields for the 1st and 2nd integral equation

$$
\begin{aligned}
\mathcal{V} * \tilde{q} - \mathcal{K} * \tilde{p} &= f_D, \quad (\mathbf{x},t) \in \Gamma_D \times (0,\infty) \\
\mathcal{D} * \tilde{p} + \mathcal{K}' * \tilde{q} &= f_N, \quad (\mathbf{x},t) \in \Gamma_N \times (0,\infty)
\end{aligned}
$$

with the right hand side

$$
\begin{aligned}
f_D &= \mathcal{C}\tilde{g}_D + \mathcal{K} * \tilde{g}_D - \mathcal{V} * \tilde{g}_N \\
f_N &= (\mathcal{I} - \mathcal{C})\,\tilde{g}_N - \mathcal{K}' * \tilde{g}_N - \mathcal{D} * \tilde{g}_D
\end{aligned}
$$

Spatial discretisation

- Geometrical approximation

$$\Gamma_h = \bigcup_{e=1}^{N_e} \tau_e$$

  τ_e denote N_e boundary elements, e.g., surface triangles
- Shape functions

$$p(\mathbf{y},t) = \sum_{i=1}^{N} p_i(t)\varphi_i(\mathbf{y}) \quad \text{and} \quad q(\mathbf{y},t) = \sum_{j=1}^{M} q_j(t)\psi_j(\mathbf{y}).$$

- Semi-discrete equations
  - Galerkin method

$$\begin{bmatrix} \mathsf{V} & -\mathsf{K} \\ \mathsf{K}^\mathsf{T} & \mathsf{D} \end{bmatrix} * \begin{bmatrix} \mathsf{q} \\ \mathsf{p} \end{bmatrix} = \begin{bmatrix} \mathsf{f}_D \\ \mathsf{f}_N \end{bmatrix}$$

  - Collocation method

$$\mathsf{V} * \mathsf{q} = \mathsf{C}\mathsf{p} + \mathsf{K} * \mathsf{p}$$

Temporal discretisation

- Calculations in frequency (Fourier) domain
  Formal transformation of the equation system $\rightsquigarrow$ frequency dependent matrices. Often only the frequency response is required in acoustics. The remaining part is to solve the equation system, e.g.,
  - GMRES
  - BiCGStab
  - for symmetric Galerkin Block-CG

  Preconditioning is essential!
- Calculation in time domain
  - Direct approach with shape functions in time and approximation of the time derivative

$$p_i(t) = \sum_{k=0}^{n} p_i^k \theta_k(t)\,, \quad q_j(t) = \sum_{k=0}^{n} q_j^k \theta_k(t) \qquad \dot{p}(\mathbf{x},t) = \frac{p(\mathbf{x},t) - p(\mathbf{x},t-\Delta t)}{\Delta t}$$

  final, recursion formula

$$\mathbf{C}_0 \mathbf{d}^m = \mathbf{D}_0 \bar{\mathbf{d}}^m + \sum_{k=1}^{m} \left(\mathbf{P}_k \mathbf{q}^{m-k} - \mathbf{Q}_k \mathbf{p}^{m-k} \right)$$

  - Convolution Quadrature Method (CQM)

CQM: Basic equations

- Equal time steps Δt
- $t_n = n \cdot \Delta t, \quad n = 0, \dots, N-1$
- Only the Laplace transform of the fundamental solution is needed

$$\left\langle (\mathcal{V} * q)(\mathbf{x}; t_n), w(\mathbf{x}) \right\rangle_\Gamma$$

$$= \sum_{i,j}^{E} \int_{\text{supp}(\varphi_i)} \varphi_i(\mathbf{x}) \int_0^{t_n} \int_{\text{supp}(\varphi_j)} G(\mathbf{x}, \mathbf{y}; t_n - \tau)\, q_j(\tau)\, \varphi_j(\mathbf{y})\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}\tau\, \mathrm{d}s_\mathbf{x}$$

$$\approx \sum_{i,j}^{E} \sum_{k=0}^{n} \omega_{ij}^{n-k}(\hat{G}, \Delta t)\, q_j(k\Delta t) = \sum_{i,j}^{E} \sum_{k=0}^{n} V_{n-k}[i,j]\, q_j(k\Delta t)$$

with $\left(s_\ell = \frac{\gamma(\zeta^\ell \mathcal{R})}{\Delta t}\,,\ \zeta = e^{\frac{2\pi \mathrm{i}}{L}}\right)$

$$\omega_{ij}^{n-k}(\hat{G}, \Delta t) = \frac{\mathcal{R}^{-(n-k)}}{L} \int\limits_{\text{supp}(\varphi_i)} \varphi_i(\mathbf{x}) \int\limits_{\text{supp}(\varphi_j)} \sum_{\ell=0}^{L-1} \hat{G}(\mathbf{x}, \mathbf{y}; s_\ell)\, \zeta^{-(n-k)\ell} \varphi_j(\mathbf{y})\, \mathrm{d}s_\mathbf{y}\, \mathrm{d}s_\mathbf{x}$$

and with $\mathcal{R} = 10^{-\frac{5}{2(N-1)}}$, $L = N - 1$,
$\gamma(z)$: characteristic function of a multistep method, e.g., a BDF2.

Rearrangement of CQM (Banjai and Sauter)

- Equation system with CQM time discretisation

$$\sum_{k=0}^{n} \frac{\mathscr{R}^{-(n-k)}}{L} \sum_{\ell=0}^{L-1} \begin{bmatrix} \hat{V}(s_\ell) & -\hat{K}(s_\ell) \\ \hat{K}^T(s_\ell) & \hat{D}(s_\ell) \end{bmatrix} \begin{bmatrix} q(k\Delta t) \\ p(k\Delta t) \end{bmatrix} \zeta^{-(n-k)\ell} = \begin{bmatrix} f_D(n\Delta t) \\ f_N(n\Delta t) \end{bmatrix}$$

- Rearrangement of the sums

$$\frac{\mathscr{R}^{-n}}{L} \sum_{\ell=0}^{L-1} \begin{bmatrix} \hat{V}(s_\ell) & -\hat{K}(s_\ell) \\ \hat{K}^T(s_\ell) & \hat{D}(s_\ell) \end{bmatrix} \zeta^{-n\ell} \sum_{k=0}^{L-1} \mathscr{R}^k \begin{bmatrix} q(k\Delta t) \\ p(k\Delta t) \end{bmatrix} \zeta^{k\ell} = \begin{bmatrix} f_D(n\Delta t) \\ f_N(n\Delta t) \end{bmatrix}$$

  with the condition

$$\omega_{-1} = \omega_{-2} = \ldots = 0 \qquad \text{and} \quad n < L-1$$

- Introduction of 'weighted' transformed variables ('weighted' FFT)

$$p_\ell^* = \sum_{k=0}^{L-1} \mathscr{R}^k p(k\Delta t) \zeta^{k\ell} \qquad q_\ell^* = \sum_{k=0}^{L-1} \mathscr{R}^k q(k\Delta t) \zeta^{k\ell}$$

  where the respective inverse operation is

$$p(n\Delta t) = \frac{\mathscr{R}^{-n}}{L} \sum_{\ell=0}^{L-1} p_\ell^* \zeta^{-n\ell} \qquad q(n\Delta t) = \frac{\mathscr{R}^{-n}}{L} \sum_{\ell=0}^{L-1} q_\ell^* \zeta^{-n\ell}$$

Decoupled problems in Laplace domain

- Calculation at 'complex frequencies' $s_\ell, \ell = 0, 1, \ldots, L-1$

$$\begin{bmatrix} \hat{V}(s_\ell) & -\hat{K}(s_\ell) \\ \hat{K}^T(s_\ell) & \hat{D}(s_\ell) \end{bmatrix} \begin{bmatrix} q^*(s_\ell) \\ p^*(s_\ell) \end{bmatrix} = \begin{bmatrix} \hat{f}_D(s_\ell) \\ \hat{f}_N(s_\ell) \end{bmatrix}$$

  with now $\hat{V} \in \mathbb{C}^{F\times F}$, $\hat{K} \in \mathbb{C}^{F\times E}$, and $\hat{D} \in \mathbb{C}^{E\times E}$
- Singular integration with integration by parts $\Rightarrow$ only weak singular integrals (formula by Erichsen and Sauter)
- Solution strategy in each frequency step
  - LDL-factorization of $\hat{V}$
  - Computation of the *Schur-Complement*

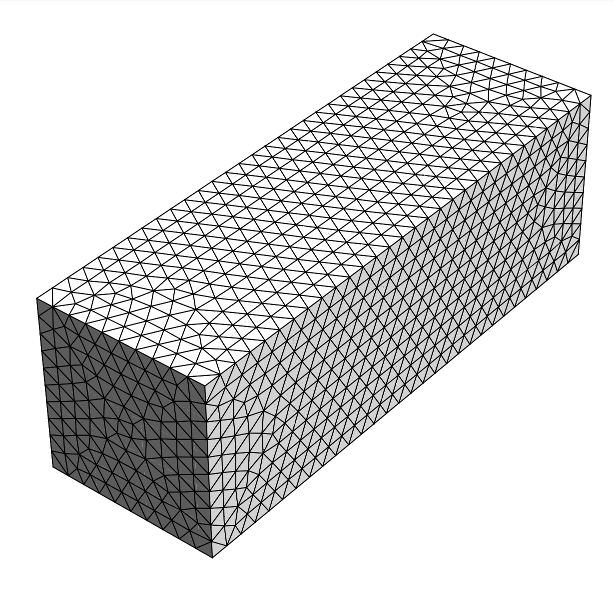
$$\hat{S} = \hat{K}^T \hat{V}^{-1} \hat{K} + \hat{D}$$

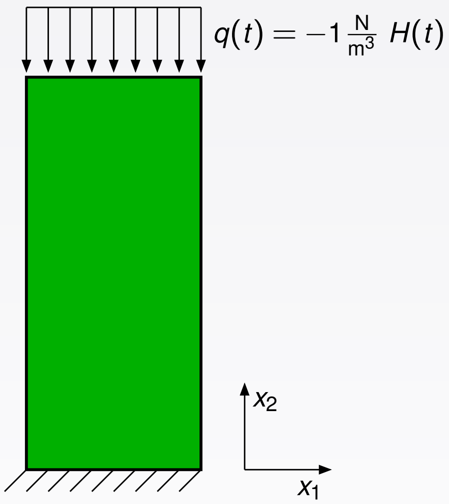
  - Determination of the displacements and tractions

$$\hat{S} p^* = \hat{f}_N - \hat{K}^T \hat{V}^{-1} \hat{f}_D$$
$$q^* = \hat{V}^{-1} \left(\hat{f}_D + \hat{K} p^* \right)$$

  - Only $L/2$ calculations are necessary due to conjugate complex frequencies s_ℓ
- Computing the time domain results

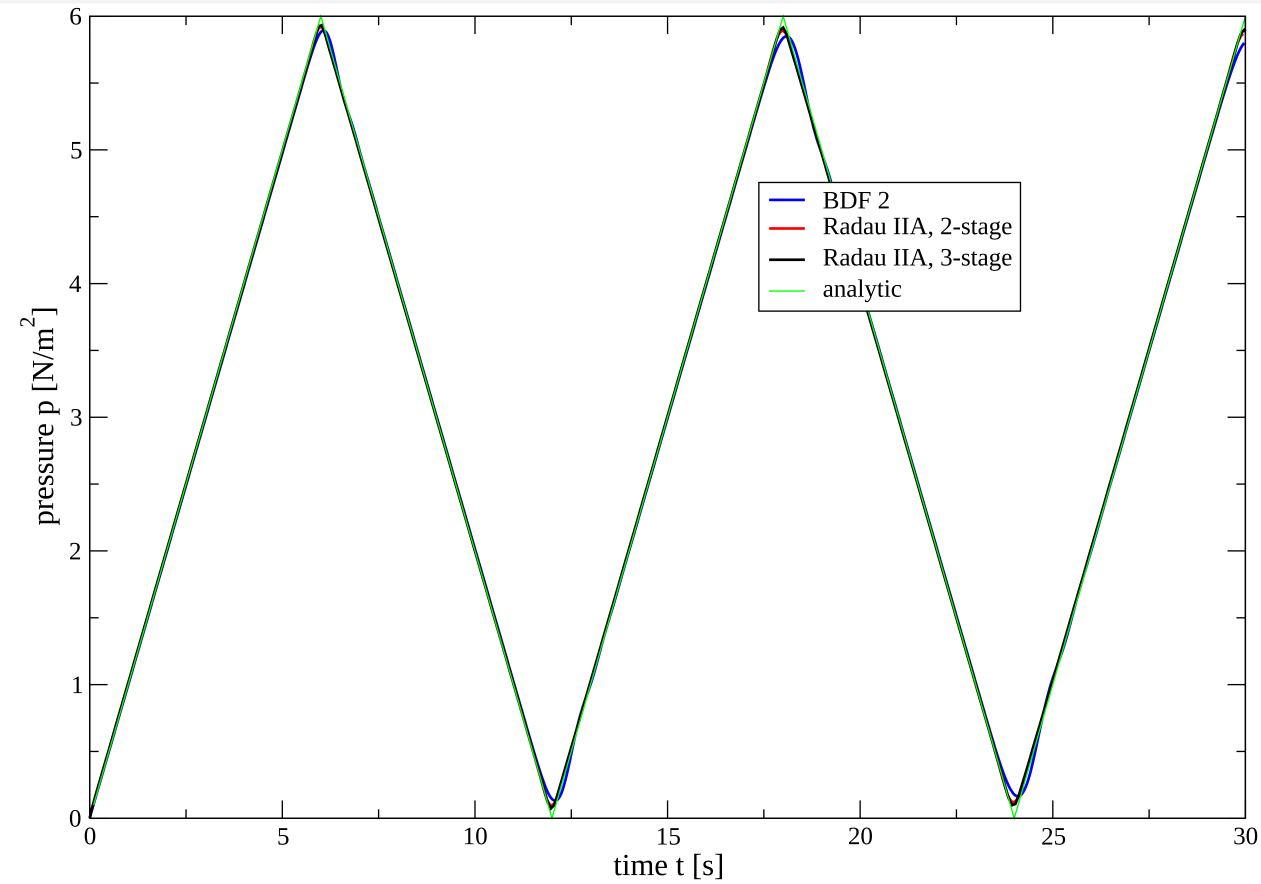
Acoustic column: Problem description



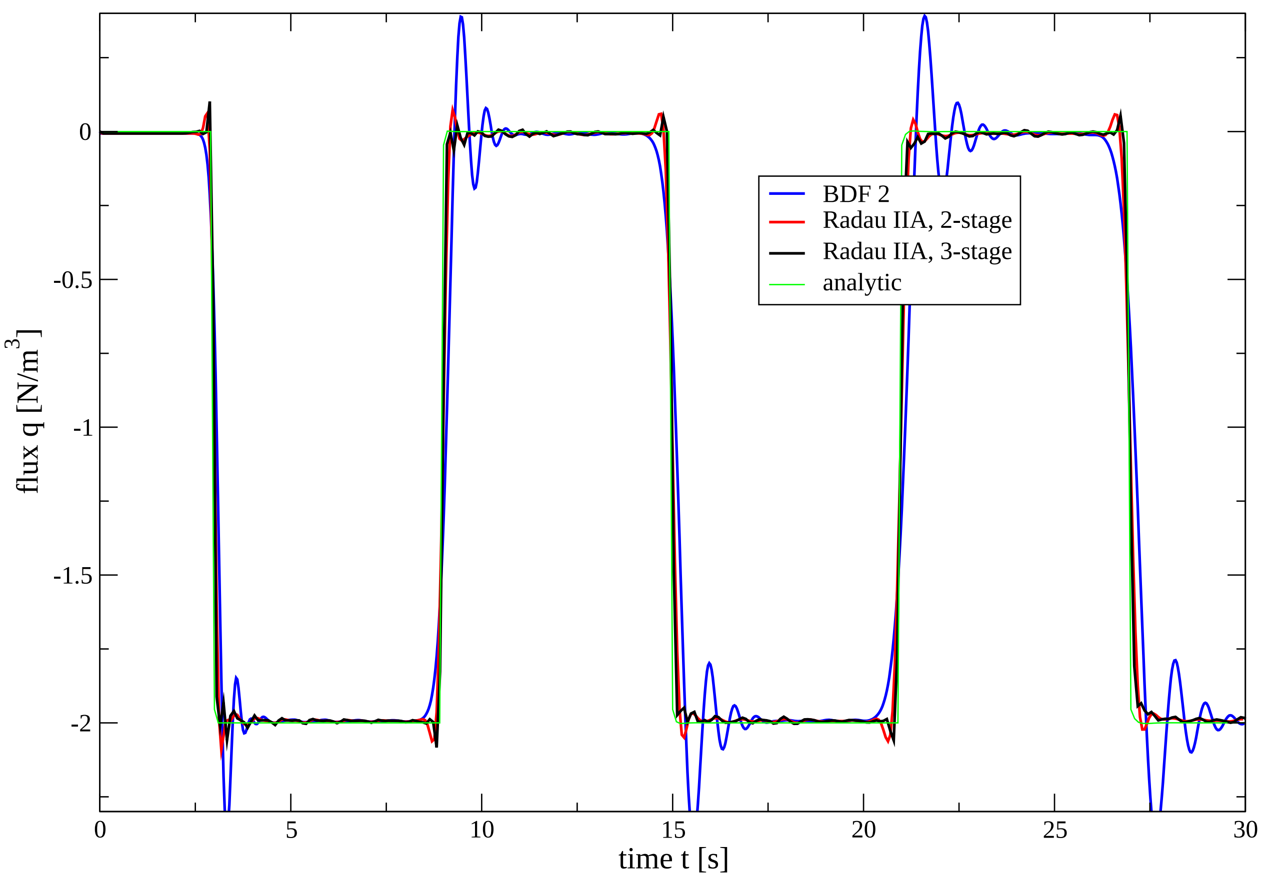


- Mesh with 3044 elements on 1524 nodes
- Shape functions: p linear and q constant
- $c = 1\,\frac{m}{s}$
- Time step size according to $\beta = \frac{c\Delta t}{r} = 0.3$
- Code used: HyENA-Library `http://www.mech.tugraz.at/HyENA`

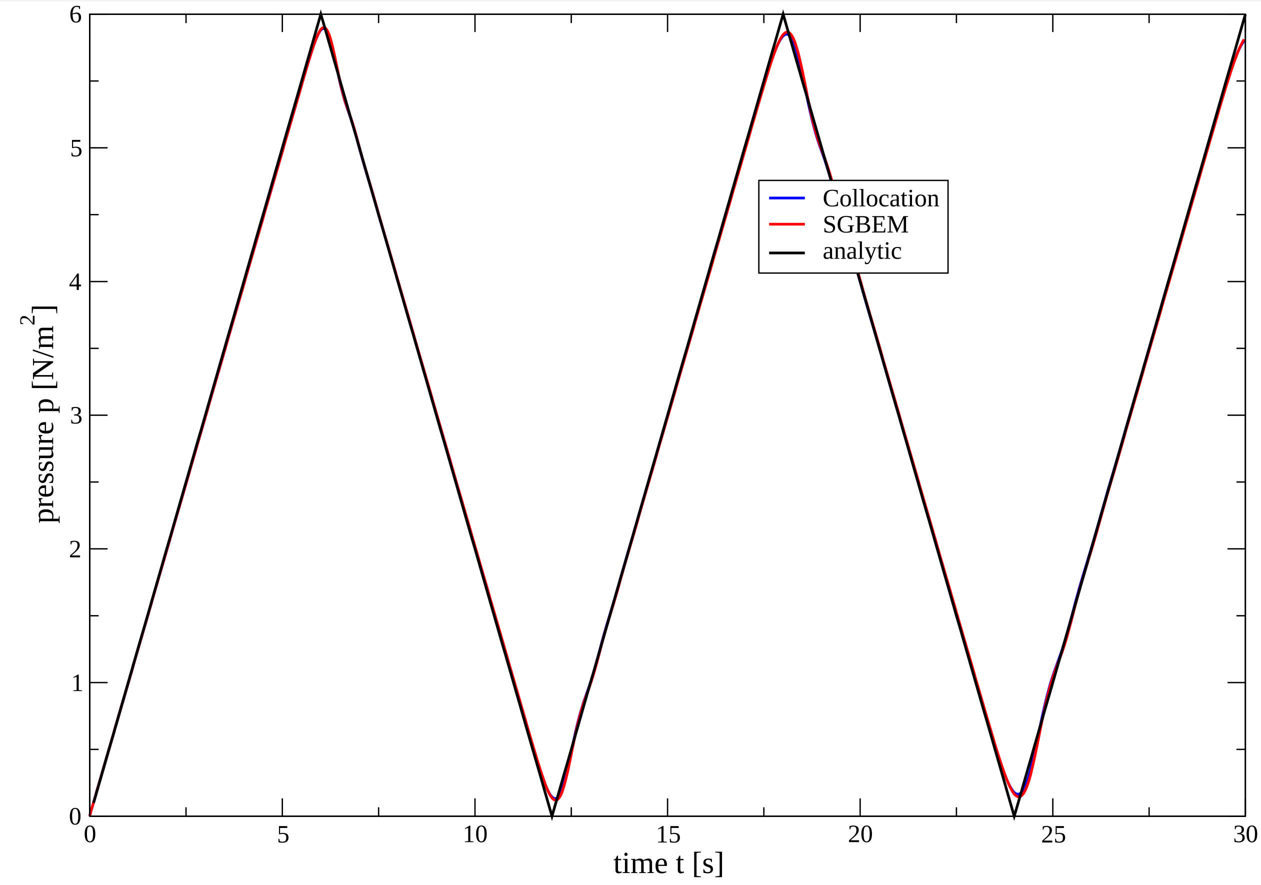
Different time discretisations



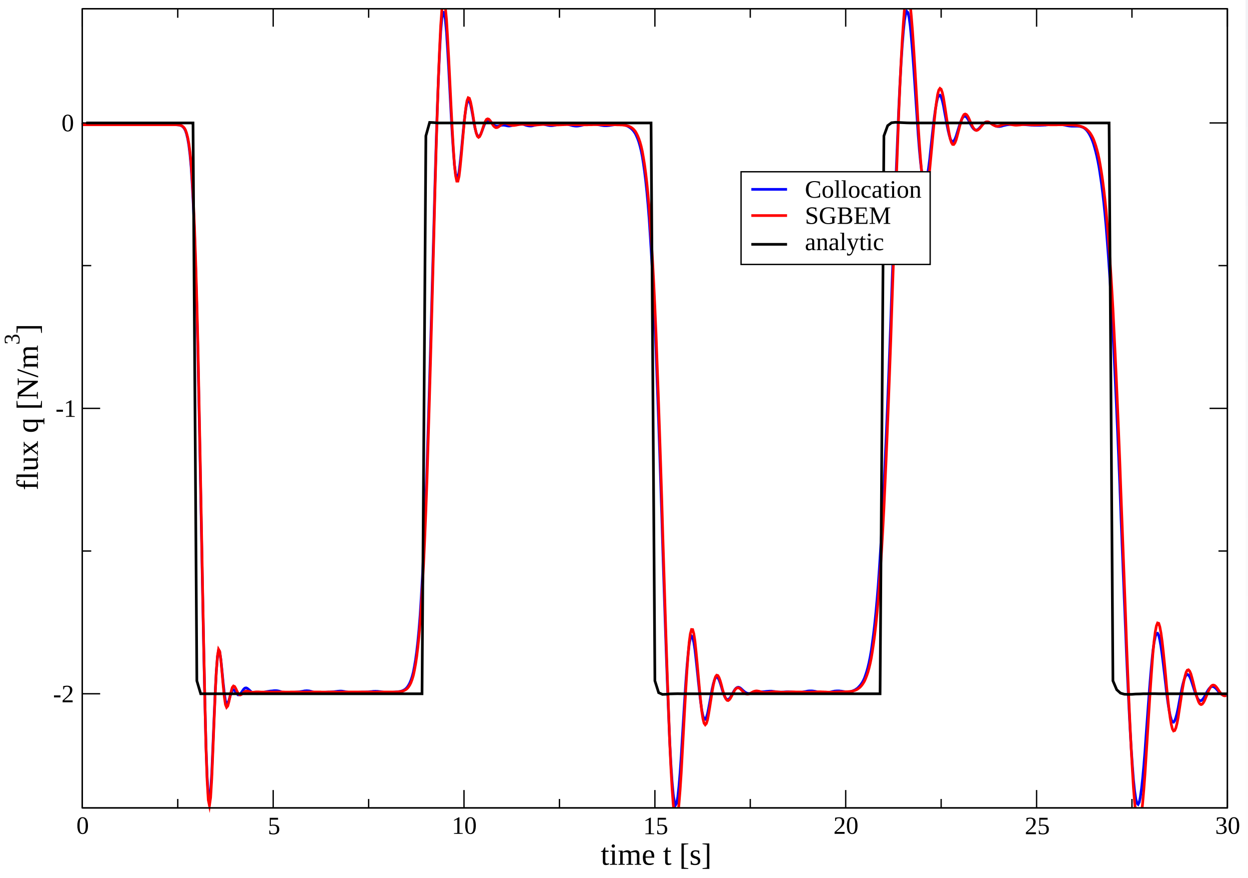
Different time discretisations



Collocations versus SGBEM



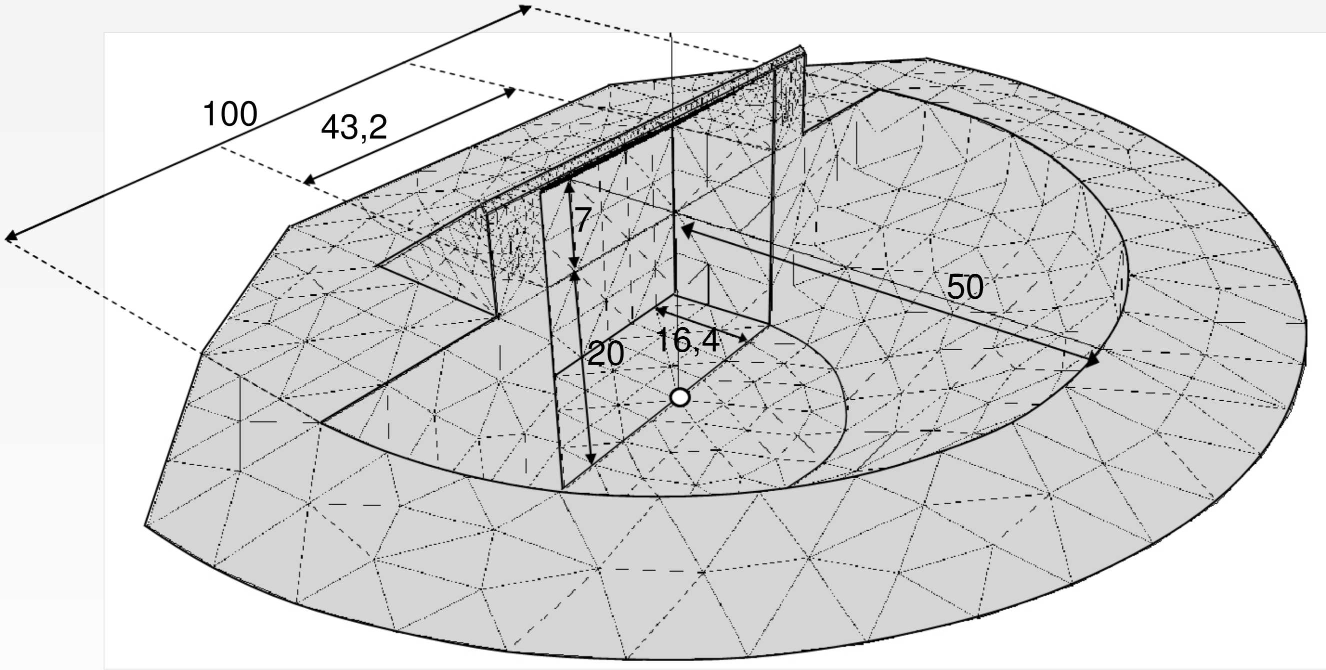
Collocations versus SGBEM



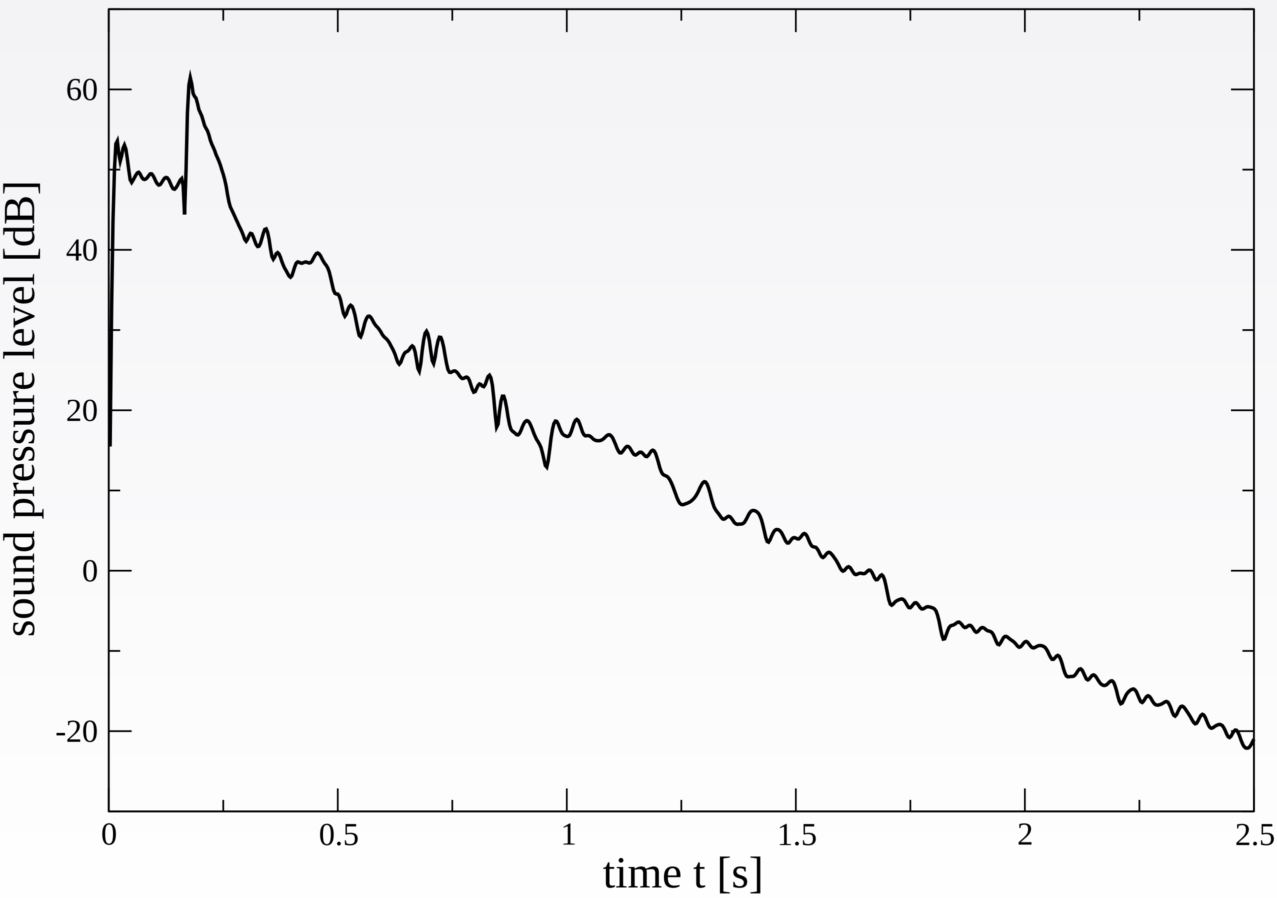
Illustrative example

[0] Arausio, The Roman Theater at Orange, France `http://www.theculturedtraveler.com/Heritage/Archives/Arausio.htm`

Sound propagation in a roman theater



Sound propagation in a roman theater



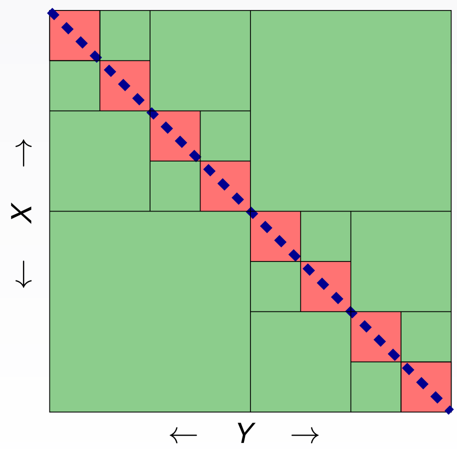
Fast BE formulations

Every fast BEM needs a proper kernel decomposition

$$k(\mathbf{x}-\mathbf{y},t-\tau) \approx k^*(\mathbf{x},\mathbf{y},t,\tau)$$

This decomposition can be done

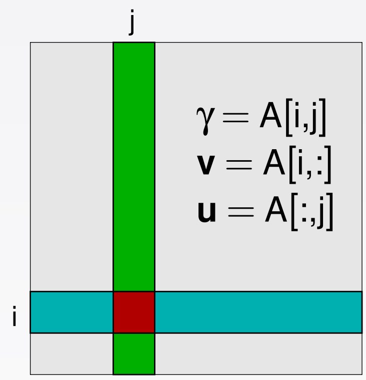
- analytically by infinite series $\rightsquigarrow$ Fast Multipole Methods
- by interpolation $\rightsquigarrow$ Panel clustering (black box technique)
- algebraically $\rightsquigarrow$ ACA
- Most algorithms are developed for the elliptic case, i.e., for the frequency domain.
- An hierarchical clustering of the matrix is necessary.



Algebraic – Low Rank Approximation

- Analytic and/or algebraic approximation techniques
  - prescribed approximation error ε
- Singular Value Decomposition – $\mathcal{O}(n^3)$
- Adaptive Cross Approximation
  - rank k approximation of A

$$S_k = \sum_{\nu=1}^{k} \gamma_\nu^{-1}\, \mathbf{u}_\nu\, \mathbf{v}_\nu^T$$



initialize $S_0 = 0,\ \ R_0 = A - S_0$
repeat
  find $\gamma_{\nu+1} = \max(R_\nu)$
  compute $\mathbf{u}_{\nu+1},\ \mathbf{v}_{\nu+1}$
  update $R_{\nu+1} = R_\nu - \gamma_{\nu+1}^{-1}\mathbf{u}_{\nu+1}\mathbf{v}_{\nu+1}^T$
  store only $\gamma_{\nu+1}, \mathbf{u}_{\nu+1}, \mathbf{v}_{\nu+1}$
until $\|R_\nu\|_F \le \varepsilon \|A\|_F$

- black box method
  - K_n itself is not touched,
  - K_n must belong to a class of asymptotically smooth functions!
- reduces storage requirement and computational time

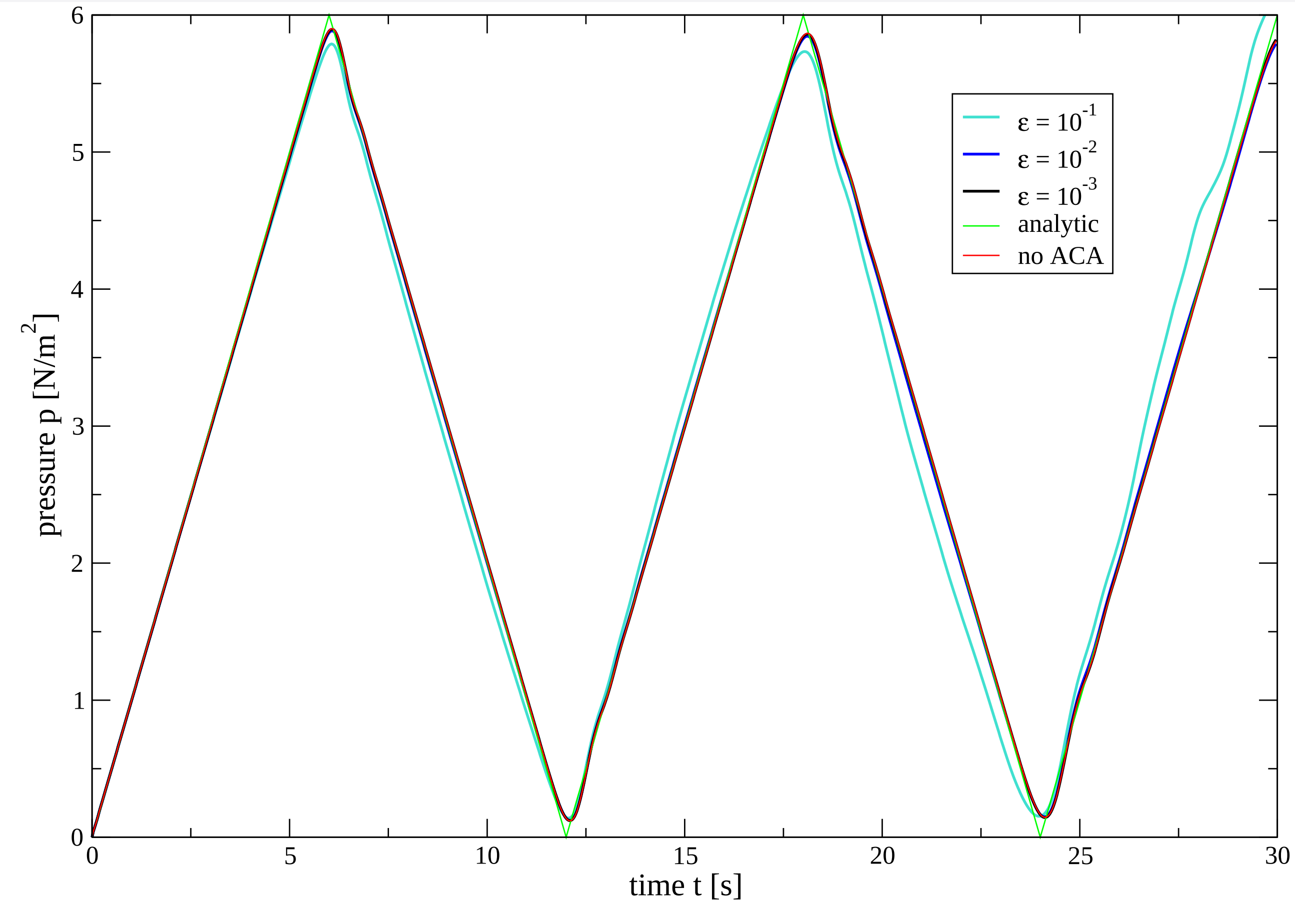
Examples for fast BE formulations

- In time domain
  - Plane wave expansion [6]
  - Panel clustering in combination with CQM [7]
    The kernel expansion is performed with Čebyšev interpolation

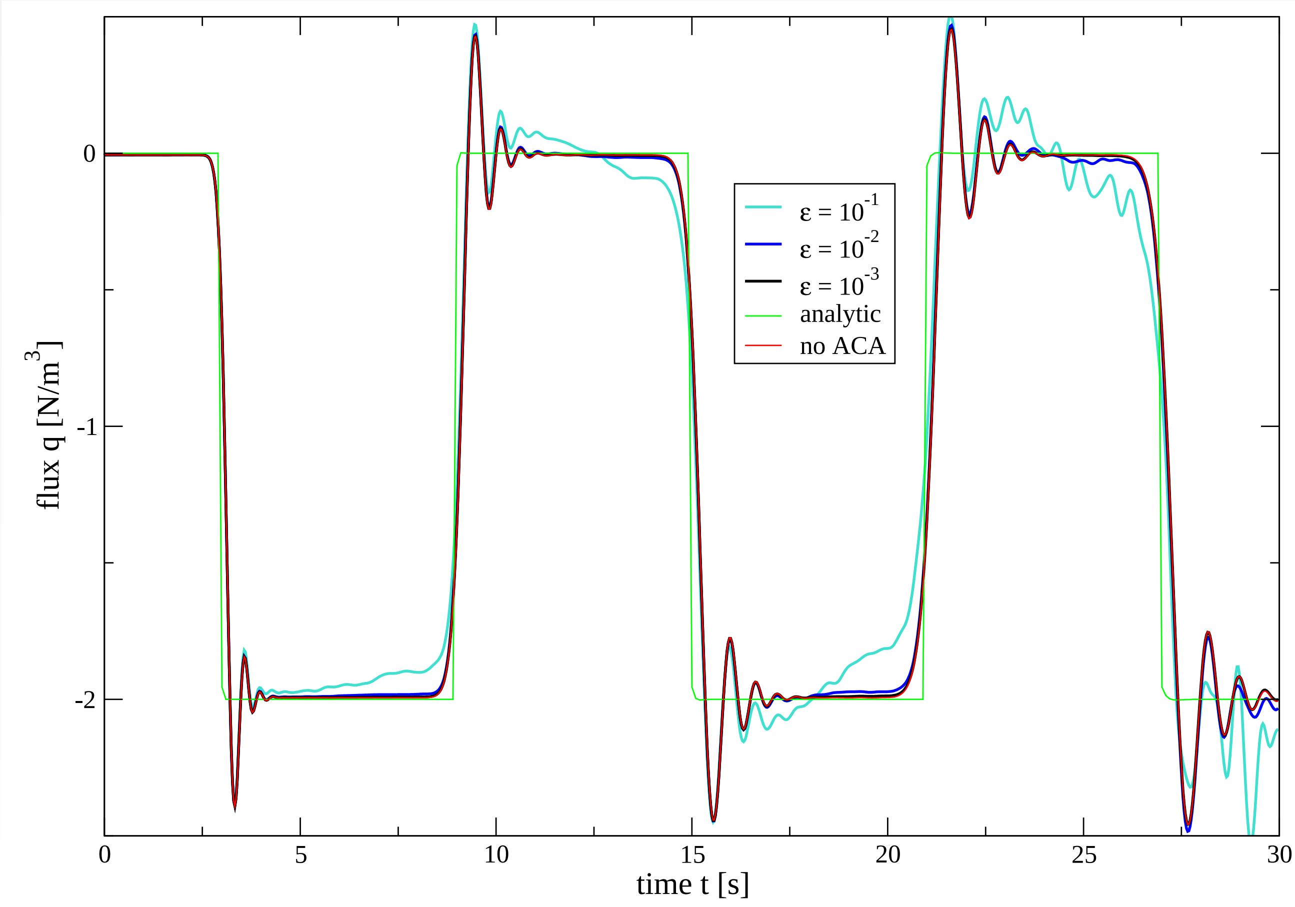
$$\omega_n^*(\mathbf{x}-\mathbf{y}) = \sum_{\mu,\nu} \mathcal{L}_c^{(\mu)}(\mathbf{x})\,\mathcal{L}_s^{(\nu)}(\mathbf{y})\,\omega_n(\mathbf{x}_\mu - \mathbf{y}_\nu)$$

  - FMM in combination with CQM [9]
  - ACA in combination with CQM in its decoupled version [8]
- In frequency domain
  - FMM with different kernel expansions for high and low frequencies [4]
  - ACA [2]
  - Panel clustering [5]

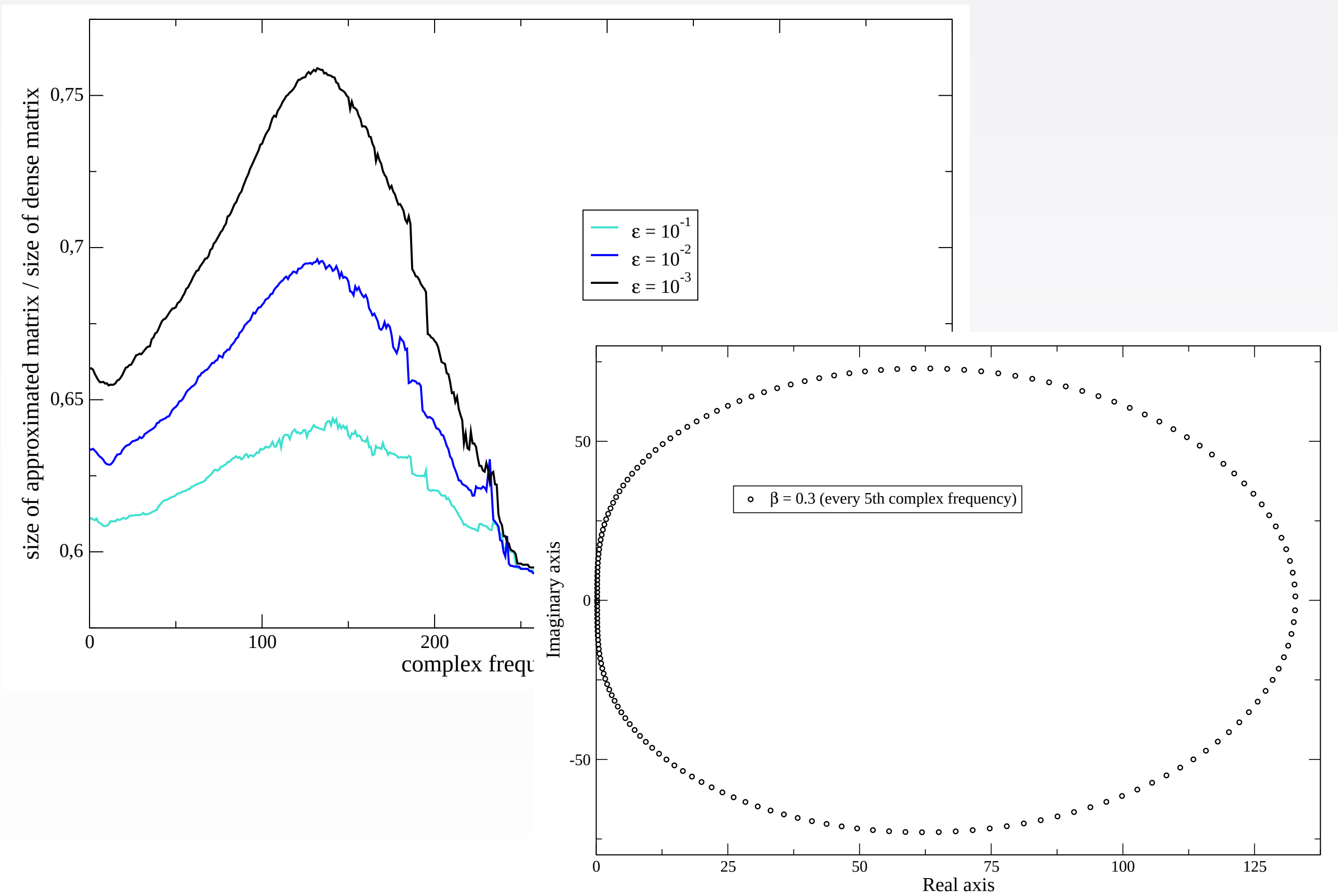
Results for the column: Pressure



Results for the column: Flux



Results for the column: Compression



Challenges and funding in Austria

Challenges

- Effective kernel expansions for higher frequencies
- Clustering for oscillatory kernels [3]
- Robust and fast solving of the equation system – construction of preconditioner without having the matrix
- Effective and robust time domain formulation [1]

Funding in Austria

- FWF (Austrian science fund)
  - 100% funding of personnel costs, 5% overhead for conferences and consumables, project specific costs
  - Only for basic research (no application)
  - $\approx 30\%$ of all proposals get funded
  - Reapplication is possible
- FFG (Austrian Research Promotion Agency)
  - Funding for research in and with industry
  - Percentage of funding is dependent on size of the company
  - Support for applicants for EU-projects
- EU-Projects

Perspective of BEM

http://www.iabem2011.it

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