**INTRODUCTION**

A. Frangi, BEM for modelling dissipation in MEMS

**BEM for modeling (dissipation) in microelectromechanical systems (MEMS)**

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DIFFERENT SOURCES OF DISSIPATION

Example: Tang resonator

solid/surface damping: internal friction, thermoelastic damping

fluid damping: rarefied regime

fluid damping: continuum regime

Q (fit)

Q (-3dB)

18kHz

Example: Tang resonator

TYPICAL APPLICATION: reduced parameter model for Q extraction

Usual assumptions:

- Only springs can deform but contribution to dissipation is negligible; shuttle is rigid (displacement denoted by \( U(t) \) w.r.t. reference to rest position)
- Small perturbations (linear response)
- Low resonating frequencies imply instantaneous fluid response

\[ \text{mass } M \quad \text{spring } K \quad \text{damper } b \]
**Typical Application:** Reduced parameter model for Q extraction

Usual assumptions:

- Only springs can deform but contribution to dissipation is negligible; shuttle is rigid (displacement denoted by $U(t)$ w.r.t. reference to rest position).
- Small perturbations (linear response).
- Low resonating frequencies imply instantaneous fluid response.

As a consequence, a quasi-static approach applies and the force exerted by the gas on the structure has the form:

$$T(x,t) = f(x)U(t)$$

and $f(x)$ is a real vector function of position.

All that is needed is a tool to estimate $f(x)$ when a unit velocity is imposed to the shuttle.

Overall gas action on structure along direction $x$ is:

$$F(t) = \left( \int_{\partial \Omega} t_x(x) dS \right) U(t)$$

Eventually:

$$B = \int_{\partial \Omega} t_x(x) dS$$

$$M\dddot{U}(t) + B\ddot{U}(t) + KU(t) = F(t)$$

Rigorous definition of Q:

$$Q = \frac{1}{2\zeta} = \frac{\omega_n}{B}$$

These assumptions lead to a quality factor $Q$ which depends on pressure only through $B$.

$Q$ crucial in gyroscopes, magnetometers, etc...

**Reduced Parameter Model**

Assumptions can be validated with phase or amplitude diagrams obtained experimentally.

Impose a sinusoidal input, measure output and phase shift between input and output.
REDUCED PARAMETER MODEL: accounting for deformability

If also springs contribute to gas damping (and more in general when an accurate estimate of $M$ is required accounting also for spring contribution), one typically assumes:

$$u(x,t) = g(x)\dot{U}(t)$$

where $g(x)$ is a typical mode of the structure (static or dynamic) and reduces to a constant on rigid shuttle. Then the Principle of Virtual Power is enforced using as virtual velocity field:

$$\tilde{u}(x) = g(x)\tilde{U}(t)$$

$$\int_{\Omega} \tilde{u}(x)\tilde{v}(x,t)d\Omega + \int_{\Omega} [\sigma(x)(x,t):\varepsilon(x,t)]d\Omega - \tilde{u}(x)\tilde{T}(x,t)ds + ... = 0$$

PVP

If $T(x,t)$ denotes the force on the structure when the velocity $g(x)$ is enforced, then the virtual power of viscous forces is:

$$\int_{\Omega} T(x,t)\tilde{v}(x,t)d\Omega$$

eventually yielding:

$$M\ddot{U}(t) + B\dot{U}(t) + KU(t) = F(t)$$

REDUCED PARAMETER MODEL: accounting for higher frequencies

If frequencies increase but linearity is preserved (almost always true the case of inertial MEMS), than one has to work in the frequency domain

$$u(x,t) = g(x)e^{it\omega}$$

If $T(x,t)$ denotes the force exerted on the structure by the fluid, then typically:

$$T(x,t) = t(x, \omega)\dot{\omega}e^{it\omega}$$

with $t(x, \omega) \in \mathbb{C}$ hence, introducing the complex damping coefficient:

$$B(\omega) = \int_{\Omega} g(x)t(x, \omega)ds$$

the final 1D model writes:

$$(-\omega^2M + i\omega B(\omega) + K)U = F$$

with $B$ contributing to both damping and stiffness.
**Knudsen Number and Flow Models**

Knudsen number $\text{Kn} = \frac{\lambda}{L}$

- $\lambda$: mean free path of molecules
- $L$: characteristic length scale

$\lambda = 0.069 \mu m$ at SATP, $\lambda \sim \frac{1}{p}$

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**Simplified Models: Linearized Reynolds Equation**

Typical of lubrication theory obtained by simply imposing the mass balance equation where $\dot{q}_x$ is the total flux along $x$ and $p$ is density (indep. of $z$)

and inserting for $\dot{q}_x$ the flux obtained from the Poiseuille parabolic velocity distribution

then linearise w.r.t. pressure: $p = p_0 + p_1$

- $h_0$ small w.r.t. plate dimensions $L$
- variations $h_1$ of $h_0$ small w.r.t. to $h_0$
- isothermal process

if this term is neglected eq. is called incompressible Reynolds


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Full Navier-Stokes model and simplifications

Reynolds number $\text{Re} = \frac{UL}{\nu}$ (U typical speed, L typical dimension, $\nu$ kinematic viscosity)
if $\text{Re} \ll 1$ neglect non-linear convective terms in Navier Stokes

Mach number $M = \frac{U}{c}$ (U typical speed, c speed of sound)
if $M \ll 1$ set $\nabla \cdot \mathbf{u} = 0$ (incompressibility) in Navier-Stokes

Stokes number $\text{St} = \frac{fL^2}{\nu}$ (f vibration frequency)
if $\text{St} \ll 1$ neglect inertia terms in Navier-Stokes

Example of biaxial accelerometer (SI units):
$L \sim 2.6 \times 10^{-6} \text{ m}; f = 4400 \text{ Hz}; \nu = 1.5 \times 10^{-5}; U \sim 2\pi f D$;
$D < \frac{1}{10} L$ (D amplitude of oscillation)
$U \sim 7 \times 10^{-3} \text{ Re} \sim 10^{-3} M \sim 7 \times 10^{-4} \text{ St} \sim 7 \times 10^{-3}$

Incompressible (quasi-static) Stokes formulation

QUASI STATIC STOKES PROBLEM

\[ \nabla p(x) - \eta \Delta u(x) = 0 \quad \nabla \cdot u(x) = 0 \quad \text{in } \Omega_2 - \Omega \]
\[ u(x) = g(x) - c_t L \hat{S}(x) \quad \text{on } S \]

$p$ pressure (defined up to a constant!)
$u$ fluid velocity,
$g$ structure velocity
$t = \sigma \cdot n$ tractions,
$t^o$ tractions projected on surface

Basic assumption:
fluid response to structure motion is instantaneous.

Time dependence?
1. inertia terms are dropped
2. velocity of structures is enforced as boundary conditions
\[ \dot{\hat{S}}(x, t) = g(x) \dot{U}(t) \]

STATIC STOKES PROBLEM WITH DIRICHLET BC
formally identical to incompressible elasticity
Comparison between full models and Reynolds

Model problem to compare Navier-Stokes, Stokes, and Reynolds solutions for damping coefficients. A 50x50x4 micron plate in motion above a 60x60x4 micron plate at 1 atm, 300K.

Comparison of Stokes, Reynolds, Navier-Stokes, solutions

STOKES PROBLEM BY BEM: MVT and slip BC

\[ \nabla p(x) - \eta \Delta u(x) = 0 \quad \nabla \cdot u(x) = 0 \quad \text{in } \Omega \]
\[ u(x) = g(x) - c_t t^S(x) \quad \text{on } S \]

\[ t \text{ tractions, } t^S \text{ tractions projected on surface} \]
\[ t^S(x) = [I - n(x) \otimes n(x)] \cdot t(x) \]
\[ c_t := \frac{2 - \sigma \lambda}{\sigma \eta} \]
\[ g(x) - c_t \frac{1}{\eta} t^S(x) = \int_{\partial \Omega} \left( \nabla(r) \cdot t(y) + c_t [K(r) \cdot n(y)] \cdot t^S(y) \right. \]
\[ \left. + \frac{1}{\eta} \left( [n(x) \cdot K(r)] \cdot t(y) - c_t [n(x) \cdot \nabla r(y) \cdot n(y)] \cdot t^S(y) \right) \right] ds_y \]

MVT: Mixed Velocity Traction. The two equations are linearly combined
STOKES PROBLEM BY BEM: MVT

$$g(x) - \frac{c_i}{2} \mathbf{t}^i(x) = \int_\gamma \left[ \mathbf{t}^i(y) \cdot \mathbf{n}(y) + c_i \left[ \mathbf{H}(y) \cdot \mathbf{n}(y) \right] \cdot \mathbf{t}^i(y) \right] \, dS_y$$

**velocity equation**

$$V_{ik}(r) = \frac{1}{8\pi \eta} \left( \frac{\delta_{ik}}{r} + \frac{r_ir_k}{r^3} \right)$$

$$K_{iqk}(r) = -\frac{3}{4\pi r^5} r_ir_q r_k$$

**traction equation**

$$\frac{1}{2} \mathbf{t}(x) = \int_\gamma \left[ -\left[ \mathbf{n}(x) \cdot \mathbf{H}(y) \right] \cdot \mathbf{t}(y) + c_i \left[ \mathbf{n}(x) \cdot \mathbf{H}(y) \cdot \mathbf{n}(y) \right] \cdot \mathbf{t}^i(y) \right] \, dS_y$$

$$W_{ijkl}(r) = \frac{\eta}{4\pi r^3} \left[ 2\delta_{ik}\delta_{lq} + \frac{3}{r^2} (\delta_{ik} r_q r_k + \delta_{iq} r_l r_k) + \delta_{ls} r_q r_k - 30 \frac{r_1 r_2 r_3 r_4}{r^4} \right]$$

very same kernels as in incompressible elasticity!!

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NULL SPACE OF VELOCITY EQUATION

**Exact null space vs numerical null space**

$$\mathbf{t}^{\alpha}(x) = \begin{cases} \mathbf{n}(x), & x \in S^\alpha \\ 0 & \text{elsewhere} \end{cases}, \quad 1 \leq \alpha \leq N$$

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WHY BEM?

- Infinite extent of air domain
- No mesh of air domain
- No unknowns in air domain
- Traction are primal variables hence very good accuracy

Largest problem dimension on a desktop
\( N_{\text{BEM}} \sim \alpha 10^6 \)

Equivalent FEM problem dimension
\( N_{\text{FEM}} \sim N_{\text{BEM}}^{3/2} \sim \beta 10^9 \)

- GMRES iterative solver
- Classical FMM with adaptive octree
- Truncation order \( p \sim 8 \)
- Computation on the fly of near integrals
- Block diagonal preconditioning
- OpenMP parallelisation

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Frangi A., Di Gioia A., Multipole BEM for the evaluation of damping forces on MEMS, Computational Mechanics, 37, 24-31 (2005)

TANG RESONATOR

- Diverge at low pressure since continuum model fails
- \( B = \int_{\Omega} t(x) dS \)

\[ M\ddot{U}(t) + B\dot{U}(t) + KU(t) = F(t) \]
BIAXIAL ACCELEROMETER

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EXPERIMENTAL VALIDATION:
LINEAR ACCELEROMETER AND SLIP B.C.

p = 1 bar

<table>
<thead>
<tr>
<th></th>
<th>Numerical “no slip”</th>
<th>Numerical “slip”</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Squeeze flow</td>
<td>$2.32 \times 10^{-4}$ N</td>
<td>$2.10 \times 10^{-4}$ N</td>
<td>-</td>
</tr>
<tr>
<td>Conette flow</td>
<td>$7.37 \times 10^{-6}$ N</td>
<td>$7.03 \times 10^{-6}$ N</td>
<td>-</td>
</tr>
<tr>
<td>Mass with holes</td>
<td>$2.10 \times 10^{-6}$ N</td>
<td>$1.94 \times 10^{-6}$ N</td>
<td>-</td>
</tr>
<tr>
<td>Total force</td>
<td>$2.41 \times 10^{-4}$ N</td>
<td>$2.19 \times 10^{-4}$ N</td>
<td>$2.21 \times 10^{-4}$ N</td>
</tr>
</tbody>
</table>

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EXPERIMENTAL VALIDATION:
LINEAR ACCELEROMETER AND SLIP B.C.

Typical comparison of simulations and results

excellent agreement at high pressure
diverge at low pressure
since continuum model fails

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Examples of full scale analysis: comb finger resonator

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Examples of full scale analysis: rotational resonator

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Examples of full scale analysis: parallel plates resonator

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EXTENSION TO HIGHER WORKING FREQUENCIES

\[ \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \eta \Delta \mathbf{u} \]

Reynolds number \( Re = \frac{UL}{\nu} \) (U typical speed, L typical dimension, \( \nu \) kinematic viscosity)
if \( Re << 1 \) neglect non-linear convective terms in Navier Stokes

Mach number \( M = \frac{U}{c} \) (U typical speed, c speed of sound)
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\( D < 1/10 L \) (D amplitude of oscillation)
\( U \sim 7 \times 10^{-3} \) \( Re \sim 10^{-3} \) \( M \sim 7 \times 10^{-4} \) \( St \sim 7 \times 10^{-3} \)

\[ \text{Incompressible frequency-domain Stokes formulation} \]

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INCOMPRESSIBLE OSCILLATORY STOKES FLOW

Mixed Velocity Traction Formulation

\[ g(x) - \frac{1}{\eta} t(x) = \int_S \left\{ \mathbf{V}(r) \cdot t(y) + \frac{\eta}{\eta} [\mathbf{n}(x) \cdot \mathbf{K}(r)] \cdot t(y) \right\} dS_y \]

Kernels involved

\[ V_k(r) = \frac{1}{8\pi \mu} A(R) \frac{\delta_k}{r} + B(R) \frac{r \delta_k}{r^3} \]

\[ K_{tk}(r) = \frac{r \delta_k}{4\pi r^3} \left[ \frac{1}{r^2} \right] - \frac{B}{2}\frac{r \delta_k}{4\pi r^3} \left[ 5B - 2e^{-R} (R + 1) \right] \]

\[ A = 2e^{-R} \left( \frac{1 + \frac{1}{R^2}}{1 + \frac{3}{R^2}} \right) - \frac{2}{R^2}, \quad B = -2e^{-R} \left( \frac{1 + \frac{3}{R^2}}{1 + \frac{3}{R^2}} \right) + \frac{6}{R^2} \]

\[ R = \sqrt{-i \frac{\alpha \rho}{\eta} \sqrt{\frac{\alpha \rho}{\eta}}} \] if \( \max(r) \sqrt{\frac{\alpha \rho}{\eta}} \ll 1 \) static limit recovered

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Wave propagation in lossy media (e.g., soils). Linear constitutive relations involve complex moduli leading to complex-valued wavenumbers with $\alpha=\pm 1, \beta<1$ (often with $\beta<<1$).

Eddy currents for the design of electrical transformers ($|\alpha|=|\beta|=1$) (Schminda et al. 2001).

Transient Stokes flow for the analysis of dissipation in Micro-Systems ($|\alpha|=|\beta|=1$) (Ye et al. 2003).

Computation of Casimir forces (attractive forces arising between uncharged conductive surfaces in vacuum) ($\alpha=0, \beta=1$) (Reid et al. 2009).

Optical tomography with $\alpha = -1; \beta > 1$. (Zacharopoulos et al. 2006).

Classical formulation with Gegenbauer addition theorem very effective if $\beta$ is large enough w.r.t. to $\alpha$ (Frangi and Bonnet, CMES, 2010).

### Example (J.White, W. Ye et al.)

- $f = 19200$ Hz

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finger gap</td>
<td>2.88 $\mu$m</td>
</tr>
<tr>
<td>Finger length</td>
<td>40.05 $\mu$m</td>
</tr>
<tr>
<td>Finger overlap</td>
<td>39.4 $\mu$m</td>
</tr>
<tr>
<td>Center plate</td>
<td>$54.9 \times 19.2$ $\mu$m$^2$</td>
</tr>
<tr>
<td>Side plate 1 : 2</td>
<td>$28.26 \times 89.6$ $\mu$m$^2$</td>
</tr>
<tr>
<td>Side plate 2 : 4</td>
<td>$11.3 \times 40.5$ $\mu$m$^2$</td>
</tr>
<tr>
<td>Thickness</td>
<td>1.96 $\mu$m</td>
</tr>
<tr>
<td>Substrate gap</td>
<td>2 $\mu$m</td>
</tr>
<tr>
<td>Truss length</td>
<td>78 $\mu$m</td>
</tr>
<tr>
<td>Truss width</td>
<td>13 $\mu$m</td>
</tr>
</tbody>
</table>

### Drag force (pN)

<table>
<thead>
<tr>
<th></th>
<th>Steady</th>
<th>Unsteady</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bottom</td>
<td>508.75</td>
<td>510.72</td>
</tr>
<tr>
<td>Side</td>
<td>284.84</td>
<td>294.50</td>
</tr>
<tr>
<td>Top</td>
<td>102.31</td>
<td>142.8</td>
</tr>
<tr>
<td>Total</td>
<td>895.9</td>
<td>948.02</td>
</tr>
</tbody>
</table>

$\sqrt{\frac{\omega \rho}{\eta}} \ll 1$
**Knudsen Number and Flow Models**

Knudsen number $\text{Kn} = \frac{\lambda}{L}$

- $\lambda$: mean free path of molecules
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$\lambda = 0.069 \, \mu m$ at SATP, $\lambda \approx \frac{1}{p}$

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**BGK Model for Boltzmann Equation**

$f(x, \xi) :$ mass density probability depends on location $x$ and molecular velocity $\xi$

- Mass density probability for a gas at rest
  
  \[ f_0 = \frac{\rho_0}{(2\pi \mathcal{R} T_0)^{3/2}} \exp \left( -\frac{|\xi|^2}{2\mathcal{R} T_0} \right) \]

BGK model

\[
\frac{\partial f}{\partial t} + \xi \cdot \nabla f - \nu(\rho, T)(f_M - f) = 0
\]

BGK model

\[
 f_M - \frac{\rho}{(2\pi \mathcal{R} T')^{3/2}} \exp \left( -\frac{|\xi - v|^2}{2\mathcal{R} T'} \right)
\]

Mass density probability for a gas at rest

\[
 f_0 = \frac{\rho_0}{(2\pi \mathcal{R} T_0)^{3/2}} \exp \left( -\frac{|\xi|^2}{2\mathcal{R} T_0} \right)
\]

Local Maxwellian
Knudsen number \( \text{Kn} = \frac{\lambda}{L} \)

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**FREE MOLECULE FLOW**

\( f(x, \xi, t) \) : mass density probability depends on location \( x \) and molecular velocity \( \xi \)

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = 0
\]

for molecules coming from other MEMS surfaces:

\[
f(x, \xi, t) = f(y, \xi, t - \frac{r}{\xi})
\]

\[
y = x + r, \quad r = -\frac{\xi}{\mathbf{v}}
\]

for molecules coming from far field region:

\[
f(x, \xi) = f_0(\xi) = \frac{\rho_0}{(2\pi R T_0)^{3/2}} \exp\left(-\frac{\xi^2}{2 R T_0}\right)
\]

Diffuse model for molecules re-emitted from surfaces

\[
f(x, \xi) = \frac{\rho_0(x)}{(2\pi R T_0(x))^{3/2}} \exp\left(\frac{|\xi - w(x)|^2}{2RT_0(x)}\right)
\]

for \( (\xi - w) \cdot \mathbf{n} > 0 \)

\[
\rho_0(x) = \left(\frac{2\pi}{R T_0(x)}\right)^{1/2} \int_{|\xi - w(x)| < 0} (\xi - w(x)) \cdot \mathbf{n(x)} |f(x, \xi)| \, d\xi
\]
ROUGHNESS OF SURFACES DUE TO ETCHING

FREE MOLECULE FLOW: LOW FREQUENCY LIMIT

If shuttle velocity is small w.r.t. thermal velocity $|\mathbf{w}| = |\mathbf{w}| / \sqrt{2RT} \ll 1$, 

$$\rho_0(x, t) \simeq \rho_0(1 + \rho_1(x, t))$$

and linearise with respect to $\rho_1$: 

$$w(x, t) = g(x)\dot{q}(t)$$

assuming 

$$\rho_1(x, t) = J(x)\dot{q}(t)$$

Limit case: 

$$\frac{\omega}{\sqrt{2RT}}L \ll 1$$

$$J(x) = \sqrt{\pi g(x) \cdot \mathbf{n}(x)}$$

$$- \frac{1}{\pi} \int_{S^+} J(y) \left( \mathbf{r} \cdot \mathbf{n}(x) \right) \left( \mathbf{r} \cdot \mathbf{n}(y) \right) \frac{1}{r^4} \, dS$$

$$+ \frac{3}{2} \frac{1}{\sqrt{\pi}} \int_{S^+} \left( \mathbf{r} \cdot \mathbf{g}(y) \right) \left( \mathbf{r} \cdot \mathbf{n}(x) \right) \left( \mathbf{r} \cdot \mathbf{n}(y) \right) \frac{1}{r^5} \, dS$$

Quasi static approximation: radiosity equation
Once $J(x, \omega)$ is available, compute tractions from similar BIE:

\[
\mathcal{F}(p) = t(x, \omega)\mathcal{F}(\dot{q})
\]

\[
- \frac{\pi^{3/2}}{\rho_0 2RT} t(x, \omega) = \left(1 + \frac{1}{2} J(x, \omega)\right) \frac{\pi^{3/2}}{2} n + \pi \tilde{g}_h(x)n(x) + \frac{\pi}{2} \tilde{g}_v(x)
- \int_{S^+} r \left( r \cdot n(x) \right) \left( r \cdot n(y) \right) \frac{1}{\rho S} J(y, \omega) T_y (\omega \tau) dS
+ 2 \int_{S^+} r \left( r \cdot \tilde{g}(y) \right) \left( r \cdot n(x) \right) \left( r \cdot n(y) \right) \frac{1}{\rho S} T_y (\omega \tau) dS
\]

\[
- \frac{\pi^{3/2}}{\rho_0 2RT} t(x, \omega) = \left(1 + \frac{1}{2} J(x, \omega)\right) \frac{\pi^{3/2}}{2} n(x) + \pi \tilde{g}_h(x)n(x) + \frac{\pi}{2} \tilde{g}_v(x)
- \frac{3 \sqrt{\pi}}{8} \int_{S^+} r \left( r \cdot n(x) \right) \left( r \cdot n(y) \right) \frac{1}{\rho S} J(y, \omega) dS
+ 2 \int_{S^+} r \left( r \cdot \tilde{g}(y) \right) \left( r \cdot n(x) \right) \left( r \cdot n(y) \right) \frac{1}{\rho S} dS
\]

\[\text{static limit case}\]

Frangi A., BEM technique for free-molecule flows in high frequency MEMS resonators, Engineering Analysis with Boundary Elements, 33, 490–498, 2009
\( \Delta q \) denotes the linear momentum change of one molecule due to one collision and \( w \) the instant velocity of the shuttle.

The total dissipation (energy transfer) induced by a single molecule before exiting the analysis domain through an in-flow surface is

\[
d_i = \sum c \Delta q \cdot w
\]

for the \( i \)-th in-flow surface the number of incoming molecules per unit time is

\[
S_i \int_{\xi > 0} \xi d\xi d\theta = S_i \frac{\rho_0}{\sqrt{4\pi}} \sqrt{\frac{2 RT_0}{\omega}}
\]

the average dissipation per unit-cycle \( D \) due to the molecules entering all the surfaces is

\[
D = \frac{\rho_0}{\sqrt{4\pi}} \sqrt{\frac{2 RT_0}{\omega}} \frac{2\pi}{\omega} \sum_i (S_i \ddot{d}_i)
\]

Finally the coefficient \( B \) in the 1D model is

\[
B = \frac{\rho_0}{\sqrt{4\pi}} \sqrt{\frac{2 RT_0}{\omega}} \frac{2}{\omega_0^2 A^2} \sum_i (S_i \ddot{d}_i)
\]

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BIE vs TEST PARTICLE MONTE CARLO METHOD

- BIE independent of velocity profile imposed
- BIE fast and robust
- BIE not affected by statistical noise
- BIE limited (so far) to diffuse reflection boundary conditions

\[
\epsilon_Q := \omega L / \sqrt{2RT} \ll 1
\]

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OUT-OF-PLANE ROTATIONAL RESONATOR

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Experimental data
Numerical

\[
\rho J \ddot{\phi}(t) + C J \dot{\phi}(t) + K \phi(t) = M(t)
\]
Qinetiq MAGNETOMETER

A. Frangi, BEM for modelling dissipation in MEMS

POLITECNICO DI MILANO

Qinetiq MAGNETOMETER

Direction of magnetic field

SEM of magnetometer structure

A. Frangi, BEM for modelling dissipation in MEMS

POLITECNICO DI MILANO
Aim:
predict the behaviour in transition regime from free-molecule flow and slip Stokes analysis.

Thank you for your attention!